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Teleportation and Superdense Coding with Genuine Quadripartite Entangled States

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Abstract

We investigate the usefulness of different classes of genuine quadripartite entangled states as quantum resources for teleportation and superdense coding. We examine the possibility of teleporting unknown one, two and three qubit states. We show that one can use the teleportation protocol to send any general one and two qubit states. A restricted class of three qubit states can also be faithfully teleported. We also explore superdense coding protocol in single-receiver and multi-receiver scenarios. We show that there exist genuine quadripartite entangled states that can be used to transmit four cbits by sending two qubits. We also discuss some interesting features of multi-receiver scenario under LOCC paradigm.

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1 Introduction

One of the intriguing feature of quantum mechanics is the quantum entanglement. This feature has been exploited to do several amazing tasks which are otherwise impossible. In particular, the entanglement can be used as a quantum resource to carry out a number of computational and information processing tasks. Such tasks include teleportation of an unknown quantum state [1], superdense coding [2], entanglement swapping [3], remote state preparation [4, 5], secret sharing [6], quantum cryptography [7] and many others. Analysis of such quantum phenomena may allow us a better understanding of the structure of the quantum mechanics framework.

A quantum system may consist of two or more subsystems which may correspondingly have bipartite or multipartite entanglement. Characterization and uses of bipartite entanglement are better understood than those of multipartite entanglement. A number of protocols which were first introduced in the context of a bipartite system can be extended to a multipartite system. However, in the case of multipartite systems, the entanglement environment is quite complex and its nature is still not fully understood. Such entangled states can be classified according to different schemes. These classes exhibit different types of entanglement properties. All classes may not be suitable for some of the information processing tasks. The tripartite states have been classified according to stochastic local operation and classical communication (SLOCC) into six categories. Two of these categories have genuine tripartite entanglement, viz. GHZ-states and W-states [8]. The utilities of these states have been explored in a number of papers [9] $-$ [20]. The quadripartite states have also been classified according to SLOCC [21]. There are nine categories. Some of them have genuine quadripartite entanglement. But usefulness of the multipartite states beyond tripartite states is still to be explored in some details. Such studies may even allow a better understanding of multiparticle entanglement and classification of quantum states according to their entanglement properties. It is worth mentioning that, a task-based classification scheme have been proposed by Bruß et al [22, 23] to classify mixed states and multipartite states according to their densecodabilty.

In this paper, we study various protocols for quantum teleportation and superdense coding in the context of quadripartite entangled states. In this scenario, there can be two, three, or four parties (Alice, Bob, Charlie and Dennis). These parties share four particles in an entangled state. We shall take these states to be genuinely quadripartite entangled states

in the sense that these states cannot be written as a direct product of bipartite entangled states, or as a direct product of a tripartite and a single-particle state.

In the next section, we enumerate the quadripartite states of qubits that we shall be considering. We shall explore the possibility of teleporting an unknown one-qubit, two-qubit, or three-qubit state. It is known (and we review it) that it is possible to teleport an unknown one-qubit state using a variety of quadripartite states. One can do this using a number of different protocols. These protocols can involve only two-particle, or three-particle or four-particle von Neumann measurements. The situation is a bit more complicated in the case of the teleportation of a general two-qubit state. Although some special two-qubit states can be teleported by a number of different quadripartite states, but a general state often cannot be. We show that a specific state can be used to teleport a general two-qubit state. This state is different from the one that was discussed in the literature [28]. A limited set of three-qubit states can also be teleported by using some of the quadripartite states which we discuss in the next section. However, to teleport an arbitrary three-qubit state, one may need an appropriate six-qubit entangled state.

Apart from the teleportation protocol, we discuss superdense coding using quadripartite states as a quantum resource. We discuss two scenarios: single-receiver and multi-receiver. In both the cases, there is just one sender. In the case of single-receiver scenario, there exist several possibilities: i) transmit two-cbits by sending one qubit, ii) transmit three, or four-cbits by sending two qubits, iii) transmit four-cbits by sending three qubits. Here the case of sending four cbits by sending two qubits is clearly more interesting. It turns out that there exist quadripartite states that can be used as a quantum resource to accomplish the task of transmitting four cbits by sending two qubits. More than four cbits cannot be sent using quadripartite entangled state because the dimensionality of the Hilbert space of four qubits is sixteen. We also discuss multi-receiver scenarios in the framework of LOCC distinguishability of a set of orthogonal states.

The plan of the paper is as follows. In section II, we enumerate the quadripartite entangled states that we consider in this paper. In section III, we discuss the use of these states as a quantum resource for the teleportation. In section IV, we discuss the protocol of superdense coding. Finally, in section V, we present our conclusions.

2 Quadripartite Entangled States

The bipartite entangled states are the simplest of the entangled states. Such states can be classified and their entanglement quantified. All bipartite entangled states belong to one equivalence class under SLOCC. The representative state of this class can be taken to be any of the Bell states. These Bell states are maximally entangled states and can be used fruitfully for the teleportation and the superdense coding as shown in the original papers that introduced these protocols [1, 2].

One may think that if we go beyond bipartite entangled states to multipartite entangled states, one may be able to accomplish tasks which are not otherwise feasible. However, the nature of entanglement in the case of multipartite entangled state is multifaceted and far from being understood. The genuine tripartite entangled states have also been classified on the basis of SLOCC [8]. There are two classes: i) the class with the representative state, $|GHZ\rangle$, ii) the class with the representative state, $|W\rangle$. States belonging to these classes can be used to successfully carry out the protocol of teleportation and superdense coding [9, 10, 11].

Beyond tripartite entangled states, one may consider quadripartite entangled states. On the basis of SLOCC, such states have also been classified. We consider a set of states from this classification and consider the possibility of implementing the protocol of teleportation and superdense coding. These states are given by

$$
\begin{aligned}
|Q1\rangle &\equiv |GHZ\rangle = \frac{1}{2}(|0000\rangle + |1111\rangle) && (1)\\
|Q2\rangle &\equiv |W\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) && (2)\\
|Q3\rangle &\equiv |\Omega\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\varphi^{+}\rangle|0\rangle + |1\rangle|\varphi^{-}\rangle|1\rangle) && (3)\\
|Q4\rangle &= \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle) && (4)\\
|Q5\rangle &= \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle) && (5)
\end{aligned}
$$

Here $|\varphi^{\pm}\rangle$ are Bell states defined in (15). According to the classification of Verstrate et al [21] quadripartite states $|GHZ\rangle$ and $|\Omega\rangle$ belong to G_{abcd} class, the states $|W\rangle, |Q4\rangle$ and $|Q5\rangle$ belong to $L_{ab_3}, L_{0_{5\oplus\bar{3}}}$ and $L_{0_{7\oplus\bar{1}}}$ respectively. Among these five entangled states $|GHZ\rangle$ and $|W\rangle$ states are symmetric with respect to permutation of particles; thus any quantum

information task performed using these states are independent of distribution of particles among the parties. For later three states it varies depending on the distribution of particles among the parties.

There are a number of ways to see that the above states have genuine quadripartite entanglement. One way is to find out the states of the system after tracing out one, two or three particles. If the state is mixed in each case, then this would be an indication of genuine quadripartite entanglement. This is what we observe. In addition we note the following. The $|GHZ\rangle$ has a mixed 3-tangle [24] of zero when one of the party is traced out. The mixed 3-tangle for $|W\rangle$ is zero and concurrence of $1/2$ when two qubits are traced out. For $|Q4\rangle$ mixed 3-tangle of $1/2$ is obtained if qubit 2,3 and 4 are traced out, zero when qubit 1 is traced out. By tracing out qubit 1 and (3 or 4) one can get a concurrence equal to $1/2$, while the other concurrence vanish. The state $|Q5\rangle$ has a concurrences equal to zero if two qubits are traced out and mixed 3-tangle of $1/2$ if particle 2,3 and 4 are traced out. The $|\Omega\rangle$ state is the cluster state introduced by Briegel and Raussendorf [31, 32] . This state is considered to have maximum connectedness and high persistence of entanglement and has been discussed extensively in the context of one-way quantum computation. The concurrence of this state is zero with any of the two qubits traced out.

3 Teleportation

In this section, we consider the teleportation of the unknown states of one, two, and three qubits. In the case of the teleportation the arbitrary state of one qubit, a number of situations may exist. There may be just two parties, or more. There is a possibility of making four-qubit, three-qubit, two-qubit, and one-qubit von Neumann measurements or a combinations of them. As making a measurement involving a larger number of qubits may be more difficult, it would be interesting to know if the protocol would work with the measurement on fewer particles. In the case of the transmission of unknown two-qubit states, there can be situations of two or three parties with quadripartite entangled states, or Alice could have option of making different types of measurements. In the case of transmitting an unknown three-qubit state, with quadripartite states as a quantum resource, there can be only two parties: Alice and Bob.

We look at some of the above situations below. Depending on the number of parties, the

classical communication cost of the protocol would be different. If there are more than two parties, then the number of transmitted cbits would increase as the information about the measurement results would be distributed. One issue in classical communication would be the number of cbits that must be transmitted to Bob who wishes to convert the state of his qubit to the state of the unknown state of the qubit that Alice has. When there are only two parties, Alice and Bob, then Alice could encode in the cbits either the results of her measurements or the unitary operations that Bob should apply to his qubit. When Alice makes a series of measurements, then the latter option is simpler. Of course, Alice and Bob would need to have a prior understanding of the option that Alice would use.

3.1 Teleportation of a single-qubit state

In this scenario, Alice wishes to teleport an unknown qubit state $|\psi\rangle_a = \alpha|0\rangle + \beta|1\rangle$ to Bob. They share a quantum channel given by one of the quadripartite states of the last section. Of the four entangled qubits, Alice has qubits 1, 2, and 3 and Bob has the qubit 4. As in the conventional teleportation protocol, Alice's strategy is to make von Neumann measurements involving particles a, 1, 2, and 3 and communicate the results to Bob. Bob then performs necessary unitary operation on his qubit according to the received message to convert the state of his qubit to that of the unknown qubit. As noted above, Alice has several choices of bases to perform the measurement. She may choose a basis of four particles or successive two-particle Bell basis or three-particle and one-particle basis for measurement. Let us now discuss various states of the last section as a quantum resource.

3.1.1 Teleportation using $|GHZ\rangle$ state

Alice has the qubits a, 1, 2, and 3. Bob has the qubit 4. Alice wishes to teleport the unknown state of the particle a,

$$|\psi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a. \tag{6}$$

Here α and β are complex numbers. The combined state of the five-qubit systems can be written as:

$$|\psi\rangle_a|GHZ\rangle_{1234} = \frac{1}{\sqrt{2}}(\alpha|0\rangle_a + \beta|1\rangle_a) \otimes (|0000\rangle_{1234} + |1111\rangle_{1234}). \tag{7}$$

We can rewrite this combined state depending on the type of the measurement Alice

wishes to make. If Alice wishes to make four-particle von Neumann measurement, then we can rewrite the above state as,

$$
\begin{aligned}
|\psi\rangle_a|GHZ\rangle_{1234} &= \frac{1}{\sqrt{2}}(\alpha|0000\rangle_{a123}|0\rangle_4 + \alpha|0111\rangle_{a123}|1\rangle_4 + \beta|1000\rangle_{a123}|0\rangle_4 + \beta|1111\rangle_{a123}|1\rangle_4) \\
&= \frac{1}{2}[|4GHZ_1^+\rangle_{a123}(\alpha|0\rangle_4 + \beta|1\rangle_4) + |4GHZ_1^-\rangle_{a123}(\alpha|0\rangle_4 - \beta|1\rangle)_4 + \\
&\quad |4GHZ_2^+\rangle_{a123}(\alpha|1\rangle_4 + \beta|0\rangle)_4 + |4GHZ_2^-\rangle_{a123}(\alpha|1\rangle_4 - \beta|0\rangle_4)], \qquad (8)
\end{aligned}
$$

where,

$$|4GHZ_1^\pm\rangle = \frac{1}{\sqrt{2}}(|0000\rangle \pm |1111\rangle) \qquad (9)$$

$$|4GHZ_2^\pm\rangle = \frac{1}{\sqrt{2}}(|0111\rangle \pm |1000\rangle). \qquad (10)$$

According to the results of the measurement, Alice sends two bits of classical information to Bob, encoding either the results of her measurements, or the unitary operation that Bob should apply. Bob performs one of the $\{\sigma_0, \sigma_1, i\sigma_2, \sigma_3\}$ operations to convert the state of his qubit to that of the unknown qubit a.

Instead of making a four-particle von Neumann measurement, Alice may wish to make a three-particle followed by one-particle von Neumann measurements. Or, there can be three parties, Alice, Bob, and Charlie. In this latter case, Alice may have the qubits a, 1, and 2, while Charlie has the qubit 3 and Bob has the qubit 4. To see how the protocol would work in this situation, we rewrite the combined state (8) as,

$$
\begin{aligned}
|\psi\rangle_a|GHZ\rangle_{1234} &= \frac{1}{\sqrt{2}}(\alpha|000\rangle_{a12}|0\rangle_3|0\rangle_4 + \alpha|011\rangle_{a12}|1\rangle_3|1\rangle_4 + \beta|100\rangle_{a12}|0\rangle_3|0\rangle_4 + \\
&\quad \beta|111\rangle_{a12}|1\rangle_3|1\rangle_4) \\
&= \frac{1}{2\sqrt{2}}(|3GHZ_1^+\rangle_{a12}|+\rangle_3 + |3GHZ_1^-\rangle_{a12}|-\rangle_3)(\alpha|0\rangle_4 + \beta|1\rangle_4) + \\
&\quad (|3GHZ_1^+\rangle_{a12}|-\rangle_3 + |3GHZ_1^-\rangle_{a12}|+\rangle_3)(\alpha|0\rangle_4 - \beta|1\rangle)_4 + \\
&\quad (|3GHZ_2^+\rangle_{a12}|+\rangle_3 - |3GHZ_2^-\rangle_{a12}|-\rangle_3)(\alpha|1\rangle_4 + \beta|0\rangle)_4 + \\
&\quad (|-3GHZ_2^+\rangle_{a12}|-\rangle_3 + |3GHZ_2^-\rangle_{a12}|+\rangle_3)(\alpha|1\rangle_4 - \beta|0\rangle_4), \qquad (11)
\end{aligned}
$$

where,

$$|3GHZ_1^\pm\rangle = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle) \qquad (12)$$

$$|3GHZ_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|011\rangle \pm |100\rangle). \tag{13}$$

and,

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \tag{14}$$

Here in the case of two parties, the simplest thing for the Alice would be to encode in two cbits the unitary operation that Bob should apply on his qubit after she makes the measurement. So here classical communication cost would still be two cbits. However, in the scenario of three parties, there will be need of three cbits of classical communication. This communication could take many forms. Some examples are: Alice sends two cbits and Charlie one cbit to Bob about the results of measurements; Charlie sends one cbit to Alice, who then sends two cbits to Bob, encoding the unitary operation that Bob should apply. In all cases Bob will make one of the $\{\sigma_0, \sigma_1, i\sigma_2, \sigma_3\}$ operations on his qubit to convert its state to that of the unknown qubit a.

Let us now consider the case, where Alice makes two successive Bell measurements, i.e., von Neumann measurements using the Bell basis:

$$\begin{aligned}
|\varphi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\
|\psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)
\end{aligned} \tag{15}$$

In the three-party scenario here, Alice will have qubits a and 1, Charlie will have the qubits 2 and 3, while Bob would have the qubit 4. In either of the two scenarios, the teleportation would be possible but with different classical communication cost. It can be seen by rewriting equation (8) as,

$$\begin{aligned}
|\psi\rangle_a|GHZ\rangle_{1234} &= \frac{1}{\sqrt{2}}(\alpha|00\rangle_{a1}|00\rangle_{23}|0\rangle_4 + \alpha|01\rangle_{a1}|11\rangle_{23}|1\rangle_4 + \\
&\quad \beta|10\rangle_{a1}|00\rangle_{23}|0\rangle_4 + \beta|11\rangle_{a1}|11\rangle_{23}|1\rangle_4) \\
&= \frac{1}{2\sqrt{2}}[(|\varphi^+\rangle_{a1}|\varphi^+\rangle_{23} + |\varphi^-\rangle_{a1}|\varphi^-\rangle_{23})(\alpha|0\rangle_4 + \beta|1\rangle_4) + \\
&\quad (|\varphi^+\rangle_{a1}|\varphi^-\rangle_{23} + |\varphi^-\rangle_{a1}|\varphi^+\rangle_{23})(\alpha|0\rangle_4 - \beta|1\rangle)_4 + \\
&\quad (|\psi^+\rangle_{a1}|\varphi^+\rangle_{23} - |\psi^-\rangle_{a1}|\varphi^-\rangle_{23})(\alpha|1\rangle_4 + \beta|0\rangle)_4 + \\
&\quad (|\psi^-\rangle_{a1}|\varphi^+\rangle_{23} - |\psi^+\rangle_{a1}|\varphi^-\rangle_{23})(\alpha|1\rangle_4 - \beta|0\rangle_4)].
\end{aligned} \tag{16}$$

Here classical communication cost would be same as in the last scenario - two cbits if there are two parties and three cbits if there are three parties.

For this entangled resource, there is one more possibility. In this case Alice first makes a Bell measurement on the particles a and 1, followed by one-particles measurement on the particles 3 and 4. Or, there could be three or four parties. The distribution of the particles in the three-party scenario will be as above. In the four-party scenario, Alice would have particles a and 1, Charlie would have particle 2 and Dennis would have particle 3, whereas Bob would have particle 4. One can easily check that in these scenarios, the teleportation is also possible because one can write $|3GHZ_1^{\pm}\rangle$ and $|3GHZ_2^{\pm}\rangle$ in (12) in terms of the Bell states (15) and the single-particle states $|\pm\rangle$ given in (14). The classical information cost would depend on the number of parties. For two parties, it would be 2 cbits; for three parties, it would be 3 cbits; whereas for four parties, it would be 4 cbits.

3.1.2 Teleportation using $|\Omega\rangle$ state

As noted earlier, this is one of the most interesting quadripartite entangled state and most powerful. As before, Alice wishes to teleport the unknown state $|\psi\rangle$ to Bob. However, the quantum resource available to her is the state $|\Omega\rangle$. This state is shared by particles 1, 2, 3, and 4. Alice has the particles 1, 2, and 3. Bob has the particle 4. The combined state of the the five particles a, 1, 2, 3, and 4 will be

$$\begin{aligned}|\psi\rangle_a|\Omega\rangle_{1234} &= (\alpha|0\rangle_a+\beta|1\rangle_a)\otimes(\frac{1}{\sqrt{2}}(|0\rangle_1|\varphi^+\rangle_{23}|0\rangle_4+|1\rangle_1|\varphi^-\rangle_{23}|1\rangle_4)\\ &= \frac{1}{\sqrt{2}}(\alpha|00\rangle_{a1}|\varphi^+\rangle_{23}|0\rangle_4+\alpha|01\rangle_{a1}|\varphi^-\rangle_{23}|1\rangle_4+\beta|10\rangle_{a1}|\varphi^+\rangle_{23}|0\rangle_4+\\ &\quad \beta|11\rangle_{a1}|\varphi^-\rangle_{23}|1\rangle_4). \end{aligned} \tag{17}$$

One can rewrite this combined state as,

$$\begin{aligned}|\psi\rangle_a|\Omega\rangle_{1234} &= |\Omega_1^+\rangle_{a123}(\alpha|0\rangle_a+\beta|1\rangle_a)+|\Omega_1^-\rangle_{a123}(\alpha|0\rangle_a-\beta|1\rangle_a)+\\ &\quad |\Omega_2^+\rangle_{a123}(\alpha|1\rangle_a+\beta|0\rangle_a)+|\Omega_2^-\rangle_{a123}(\alpha|1\rangle_a-\beta|0\rangle_a). \end{aligned} \tag{18}$$

Here the basis vectors are:

$$|\Omega_1^{\pm}\rangle = \frac{1}{\sqrt{2}}|00\rangle|\varphi^+\rangle \pm |11\rangle|\varphi^-\rangle,$$

$$|\Omega_2^{\pm}\rangle = \frac{1}{\sqrt{2}}|01\rangle|\varphi^-\rangle \pm |10\rangle|\varphi^+\rangle. \quad (19)$$

After making the measurement in this basis, Alice can convey her results to Bob using two classical bits. Bob then can apply appropriate unitary transformation to his qubit, as in the case of the GHZ-state, and complete the protocol. Interestingly, there exist another set of four-particle basis vectors that Alice can use to make measurement. This basis, GHZ-basis, has in addition to the states (9) and (10), following states

$$|4GHZ_3^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0011\rangle \pm |1100\rangle), \quad (20)$$

$$|4GHZ_4^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0100\rangle \pm |1011\rangle). \quad (21)$$

In this GHZ-basis, the combined state (17) can be written as,

$$\begin{aligned}|\psi\rangle_a|\Omega\rangle_{1234} = \frac{1}{2\sqrt{2}}[&(|4GHZ_1^+\rangle_{a123} + |4GHZ_3^-\rangle_{a123})(\alpha|0\rangle_4 - \beta|1\rangle_4) + \\ &(|4GHZ_1^-\rangle_{a123} + |4GHZ_3^+\rangle_{a123})(\alpha|0\rangle_4 + \beta|1\rangle_4) + \\ &(-|4GHZ_2^-\rangle_{a123} + |4GHZ_4^+\rangle_{a123})(\alpha|1\rangle_4 + \beta|0\rangle_4) + \\ &(-|4GHZ_2^+\rangle_{a123} + |4GHZ_4^-\rangle_{a123})(\alpha|1\rangle_4 - \beta|0\rangle_4)]. \quad (22)\end{aligned}$$

By encoding the unitary operations in the two cbits, Alice can convey the information to Bob who can complete the protocol.

Let us now consider the next scenario where Alice makes a three-particle von Neumann measurement followed by a one-particle measurement. As in the case of $|GHZ\rangle$, there can be two situations. There can be two parties or three parties. In this case we can rewrite (17) as,

$$\begin{aligned}|\psi\rangle_a|\Omega\rangle_{1234} &= (\alpha|0\rangle_a + \beta|1\rangle_a) \otimes (\frac{1}{\sqrt{2}}(|0\rangle_1|\varphi^+\rangle_{23}|0\rangle_4 + |1\rangle_1|\varphi^-\rangle_{23}|1\rangle_4) \\ &= \frac{1}{4}[(|3GHZ_1^+\rangle_{a12}|-\rangle_3 + |3GHZ_1^-\rangle_{a12}|+\rangle_3 + |3GHZ_3^+\rangle_{a12}|+\rangle_3 - |3GHZ_3^-\rangle_{a12}|-\rangle_3) \\ &\qquad (\alpha|0\rangle_4 + \beta|1\rangle_4) + \\ &\quad (|3GHZ_1^+\rangle_{a12}|+\rangle_3 + |3GHZ_1^-\rangle_{a12}|-\rangle_3 - |3GHZ_3^+\rangle_{a12}|-\rangle_3 + |3GHZ_3^-\rangle_{a12}|+\rangle_3) \\ &\qquad (\alpha|0\rangle_4 - \beta|1\rangle)_4 + \\ &\quad (|3GHZ_4^+\rangle_{a12}|+\rangle_3 + |3GHZ_4^-\rangle_{a12}|-\rangle_3 + |3GHZ_2^+\rangle_{a12}|-\rangle_3 - |3GHZ_2^-\rangle_{a12}|+\rangle_3)\end{aligned}$$

$$
\begin{aligned}
&(\alpha|1\rangle_4 + \beta|0\rangle)_4 + \\
&(|3GHZ_4^+\rangle_{a12}|-\rangle_3 + |3GHZ_4^-\rangle_{a12}|+\rangle_3 - |3GHZ_2^+\rangle_{a12}|+\rangle_3 + |3GHZ_2^-\rangle_{a12}|-\rangle_3) \\
&(\alpha|1\rangle_4 - \beta|0\rangle_4)]. \qquad (23)
\end{aligned}
$$

Here one can use three-particle GHZ-basis. It has in addition to the states given in (12) and (13), we have the following states,

$$
\begin{aligned}
|3GHZ_3^\pm\rangle &= \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \\
|3GHZ_4^\pm\rangle &= \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle). \qquad (24)
\end{aligned}
$$

As in the case of $|GHZ\rangle$ state, in the two-party situation, Alice needs to send two cbits to Bob (e.g., encoding the four unitary operations). In the three-party situation, combined classical information cost will be three cbits. As before, this classical communication could take many forms. For example, Charlie can send one cbit of information about his measurement to Alice. On the basis of her results, Alice can send two cbits of information to Bob, encoding the unitary transformation. After receiving the classical communication, Bob can complete the protocol by applying a suitable unitary operation.

The strategy of making two successive Bell measurements also works if we make measurements on suitably chosen qubits. (This is because, this state is not symmetric under the permutation of qubits.) If Alice makes a measurement on qubits 'a2' and '13', then we can write the combined state as:

$$
\begin{aligned}
|\psi\rangle_a|\Omega\rangle_{1234} &= \frac{1}{\sqrt{2}}(\alpha|0\rangle_a + \beta|1\rangle_a)(|0\rangle_1|\varphi^+\rangle_{23}|0\rangle_4 + |1\rangle_1|\varphi^-\rangle_{23}|1\rangle_4) \\
&= \frac{1}{4}[(|\varphi^+\rangle_{a2}|\varphi^+\rangle_{13} + |\varphi^-\rangle_a|\varphi^-\rangle_{13} + |\psi^+\rangle_{a2}|\psi^-\rangle_{13} + |\psi^-\rangle_{a2}|\psi^+\rangle_{13})(\alpha|0\rangle_4 - \beta|1\rangle_4) + \\
&\quad (|\varphi^+\rangle_{a2}|\varphi^-\rangle_{13} + |\varphi^-\rangle_a|\varphi^+\rangle_{13} + |\psi^+\rangle_{a2}|\psi^+\rangle_{13} + |\psi^-\rangle_{a2}|\psi^-\rangle_{13})(\alpha|0\rangle_4 + \beta|1\rangle_4) + \\
&\quad (|\varphi^+\rangle_{a2}|\psi^+\rangle_{13} - |\varphi^-\rangle_a|\psi^-\rangle_{13} + |\psi^+\rangle_{a2}|\varphi^-\rangle_{13} - |\psi^-\rangle_{a2}|\varphi^+\rangle_{13})(\alpha|1\rangle_4 + \beta|0\rangle_4) - \\
&\quad (|\varphi^+\rangle_{a2}|\psi^-\rangle_{13} - |\varphi^-\rangle_a|\psi^+\rangle_{13} - |\psi^+\rangle_{a2}|\varphi^+\rangle_{13} + |\psi^-\rangle_{a2}|\varphi^+\rangle_{13})(\alpha|1\rangle_4 - \beta|0\rangle_4)]. \qquad (25)
\end{aligned}
$$

Here the classical communication cost will be two cbits for two-party situation and four cbits in the three-party situation.

Other situation is a Bell measurement followed by two one-particle measurements. As before, one could have two-party, three-party and four-party situations. The protocol would work as before with appropriate classical communication cost. This is because one can rewrite $|3GHZ_n^{\pm}\rangle$ $(n = 1 - 4)$ in (23) in terms of the single-particle states (14) and the Bell states (15).

3.1.3 Teleportation using $|W\rangle$ state

Next we consider the W-state given in (2). This state does not allow the faithful teleportation of an unknown qubit state. However, a modified version of this state that also belongs to the W-state category under the SLOCC classification can work. This is shown below. The distribution of the four qubits is as earlier. The combined state of the particle a, 1, 2, 3, and 4 can be written as

$$\begin{aligned}
|\psi\rangle_a|W\rangle_{1234} &= (\alpha|0\rangle_a + \beta|1\rangle_a) \otimes \frac{1}{2}(|1000\rangle_{1234} + |0100\rangle_{1234} + |0010\rangle_{1234} + |0001\rangle_{1234}) \\
&= \frac{1}{2}(\alpha|0100\rangle_{a123}|0\rangle_4 + \alpha|0010\rangle_{a123}|0\rangle_4 + \alpha|0001\rangle_{a123}|0\rangle_4 + \alpha|0000\rangle_{a123}|1\rangle_4 + \\
&\quad \beta|1100\rangle_{a123}|0\rangle_4 + \beta|1010\rangle_{a123}|0\rangle_4 + \beta|1001\rangle_{a123}|0\rangle_4 + \beta|1000\rangle_{a123}|1\rangle_4). \quad (26)
\end{aligned}$$

It does not seem possible to rewrite the above state to teleport the qubit faithfully. In the case of three-qubit W-state, it was shown[11] that instead of the W state, one needs to consider the state $|W_n\rangle$, which is $\frac{1}{\sqrt{2+2n}}(|100\rangle + \sqrt{n}|010\rangle + \sqrt{n+1}|001\rangle)$. This state can be used for the perfect teleportation of an unknown qubit. Analogously, one could construct the state for the case of four qubits. We can consider the state

$$|W_{mn}\rangle = \frac{1}{\sqrt{2m+2n+2}}(|1000\rangle + me^{i\rho}|0100\rangle + ne^{i\eta}|0010\rangle + \sqrt{m+n+1}e^{i\sigma}|0001\rangle). \quad (27)$$

Here m and n are real numbers. For simplicity, one could set the phases to unity and choose $m = n = 1$,

$$|W_{11}\rangle = \frac{1}{\sqrt{6}}(|1000\rangle + |0100\rangle + |0010\rangle + \sqrt{3}|0001\rangle. \quad (28)$$

With this quantum resource, the combined state of five particles would be

$$|\psi\rangle_a|W_{11}\rangle_{1234} = \frac{1}{\sqrt{6}}(\alpha|0\rangle_a + \beta|1\rangle_a) \otimes (|1000\rangle_{1234} + |0100\rangle_{1234} + |0010\rangle_{1234} + \sqrt{3}|0001\rangle_{1234})$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{6}}(\alpha|0100\rangle_{a123}|0\rangle_4 + \alpha|0010\rangle_{a123}|0\rangle_4 + \alpha|0001\rangle_{a123}|0\rangle_4 + \sqrt{3}\alpha|0000\rangle_{a123}|1\rangle_4 + \\
&\quad \beta|1100\rangle_{a123}|0\rangle_4 + \beta|1010\rangle_{a123}|0\rangle_4 + \beta|1001\rangle_{a123}|0\rangle_4 + \sqrt{3}\beta|1000\rangle_{a123}|1\rangle_4) \\
&= \frac{1}{\sqrt{6}}[|\eta^+\rangle_{a123}(\alpha|0\rangle_4 + \beta|1\rangle_4) + |\eta^-\rangle_{a123}(\alpha|0\rangle_4 - \beta|1\rangle_4) + \\
&\quad |\zeta^+\rangle_{a123}(\beta|0\rangle_4 + \alpha|1\rangle_4) + |\zeta^-\rangle_{a123}(\alpha|0\rangle_4 - \beta|1\rangle_4, \qquad (29)
\end{aligned}
$$

where,

$$
\begin{aligned}
|\eta^\pm\rangle &= \frac{1}{\sqrt{6}}(|0100\rangle + |0010\rangle + |0001\rangle \pm \sqrt{3}|1000\rangle), \\
|\zeta^\pm\rangle &= \frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |1001\rangle \pm \sqrt{3}|0000\rangle). \qquad (30)
\end{aligned}
$$

Now Alice can send the two classical bits of information to Bob to inform him about the results of his measurement, or the unitary operation that he should apply. Bob then completes the protocol by applying appropriate unitary transformation.

One can make a more general observation about constructing a suitable state. This state can be $p|1000\rangle + q|0100\rangle + r|0010\rangle + s|0001\rangle$ where $|p|^2 + |q|^2 + |r|^2 = |s|^2$ which is suitable for teleportation of qubit. For a more general case of N-qubits the suitable state would be $a_1|10...0\rangle + a_2|010...0\rangle + ... + a_N|00....1\rangle$ with coefficients satisfying $|a_1|^2 + |a_2|^2 + ... + |a_{N-1}|^2 = |a_N|^2$.

As earlier, there also exist scenarios of Alice making three-particle measurement followed by one-particle measurement; two successive Bell measurements; one Bell measurement followed by two one-particle measurements. In all of these scenarios, there could exist multi-party situations. These scenarios may work out with suitable W-class state. As before, more parties would mean more classical communication cost.

3.1.4 Teleportaion using $|Q4\rangle$ state

The state $|Q4\rangle$ can also be used to teleport an unknown state $|\psi\rangle$. Unlike the GHZ-state, here the state changes with permutation of the particles. The states obtained on permutation would also belong to the same SLOCC class. However, for different states, one would need different distribution of particles for the measurement. If the particles are distributes such that particles a, 1, 2, and 3 are with Alice and 4 with Bob then this state cannot be used for teleportation; but if the distribution is such that Alice has particles a, 1, 3, 4, and Bob

has particle 2, it would lead to faithful teleportation. We can see how the protocol works as follows. The combined state of the five particles can be written as

$$
\begin{aligned}
|\psi\rangle_a|Q4\rangle_{1234} &= \frac{1}{2}(\alpha|0\rangle_a + \beta|1\rangle_a)(|0000\rangle_{1234} + |0101\rangle_{1234} + |1000\rangle_{1234} + |1110\rangle_{1234} \\
&= \frac{1}{2}[\alpha|0000\rangle_{a134}|0\rangle_2 + \alpha|0001\rangle_{a134}|1\rangle_2 + \alpha|0100\rangle_{a134}|0\rangle_2 + \alpha|0011\rangle_{a134}|1\rangle_2 + \\
&\quad \beta|1000\rangle_{a134}|0\rangle_2 + \beta|1001\rangle_{a134}|1\rangle_2 + \beta|1100\rangle_{a134}|0\rangle_2 + \beta|1011\rangle_{a134}|1\rangle_2]. \quad (31)
\end{aligned}
$$

Now Alice can use one of the following set of basis vectors to make four-particle von-Neumann measurements. One set of basis vectors are

$$
|\rho_1^{\pm}\rangle = \frac{1}{2}[(|0000\rangle + |0100\rangle) \pm (|1001\rangle + |1011\rangle)] \quad (32)
$$

$$
|\rho_2^{\pm}\rangle = \frac{1}{2}[(|0001\rangle + |0011\rangle) \pm (|1001\rangle + |1100\rangle)], \quad (33)
$$

while the other set is,

$$
|\tau_1^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle \pm |1001\rangle) \quad (34)
$$

$$
|\tau_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0001\rangle \pm |1000\rangle) \quad (35)
$$

$$
|\tau_3^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0100\rangle \pm |1011\rangle) \quad (36)
$$

$$
|\tau_4^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0011\rangle \pm |1100\rangle). \quad (37)
$$

Using the basis (32)-(33), one can rewrite (31) as

$$
\begin{aligned}
|\psi\rangle_a|Q4\rangle_{1234} &= \frac{1}{2}[|\rho_1^+\rangle_{a134}(\alpha|0\rangle_2 + \beta|1\rangle_2) + |\rho_1^-\rangle_{a134}(\alpha|0\rangle_2 - \beta|1\rangle_2) + \\
&\quad |\rho_2^+\rangle_{a134}(\alpha|1\rangle_2 + \beta|0\rangle_2) + |\rho_2^-\rangle_{a134}(\alpha|1\rangle_2 - \beta|0\rangle_2)], \quad (38)
\end{aligned}
$$

while using the basis (34)-(37), we can rewrite (31) as,

$$
\begin{aligned}
|\psi\rangle_a|Q4\rangle_{1234} &= \frac{1}{2\sqrt{2}}[(|\tau_1^+\rangle + |\tau_3^+\rangle)(\alpha|0\rangle + \beta|1\rangle) + (|\tau_1^-\rangle + |\tau_3^-\rangle)(\alpha|0\rangle - \beta|1\rangle) + \\
&\quad (|\tau_2^+\rangle + |\tau_4^+\rangle)(\alpha|1\rangle + \beta|0\rangle) + (|\tau_2^-\rangle + |\tau_4^-\rangle)(\alpha|1\rangle - \beta|0\rangle)]. \quad (39)
\end{aligned}
$$

Irrespective of the basis set Alice uses, she needs to send only two classical bits of information to Bob. Then, Bob can apply suitable unitary operator to convert the state of his qubit to that of (7).

It is interesting to note that if the particles are distributed such that Alice has particles a, 1, 2, 3 and Bob has 4, then these exists a state in this SLOCC class,

$$|Q4_{11}\rangle = \frac{1}{\sqrt{6}}(|0000\rangle + |1000\rangle + |1110\rangle + \sqrt{3}|0101\rangle, \tag{40}$$

which can be used for the teleportation if the measurement is performed in the basis,

$$\begin{aligned} |\eta\rangle^{\pm} &= \frac{1}{\sqrt{6}}(|0000\rangle + |0100\rangle + |0111\rangle \pm \sqrt{3}|1010\rangle) & (41) \\ |\zeta\rangle^{\pm} &= \frac{1}{\sqrt{6}}(|1000\rangle + |1100\rangle + |1111\rangle \pm \sqrt{3}|0010\rangle). & (42) \end{aligned}$$

As earlier, there exist the scenarios where Alice chooses to make a three-particle measurement followed by a one-particle measurement; or she makes two successive Bell measurements; or she makes a Bell measurement followed by two one-particle measurements. These scenarios could have multiparty situations. One needs to investigate further whether these scenarios could be realized with this $|Q4\rangle$ state. One can also explore other states of this class for their suitability for realizing various scenarios.

3.1.5 Teleportation using $|Q5\rangle$ state

This entangled state can also be used as a suitable quantum resource. The distribution of the four qubits is as before. The combined state of the particle a, 1, 2, 3, and 4 can be written as:

$$\begin{aligned} |\psi\rangle_a|Q5\rangle_{1234} &= (\alpha|0\rangle_a + \beta|1\rangle_a) \otimes \frac{1}{2}(|0000\rangle_{1234} + |1011\rangle_{1234} + |1101\rangle_{1234} + |1110\rangle_{1234}) \\ &= \frac{1}{2}(\alpha|0000\rangle_{a123}|0\rangle_4 + \alpha|0101\rangle_{a123}|1\rangle_4 + \alpha|0110\rangle_{a123}|1\rangle_4 + \alpha|0111\rangle_{a123}|0\rangle_4 + \\ &\quad \beta|1000\rangle_{a123}|0\rangle_4 + \beta|1101\rangle_{a123}|1\rangle_4 + \beta|1110\rangle_{a123}|1\rangle_4 + \beta|1111\rangle_{a123}|0\rangle_4). \end{aligned} \tag{43}$$

It turns out that one can teleport the state $|\psi\rangle$, if Alice makes a four-particle von Neumann measurement using at least two different sets of basis vectors. If Alice uses the following set of basis vectors:

$$|\varphi_1^{\pm}\rangle = \frac{1}{2}[(|0000\rangle + |0111\rangle) \pm (|1101\rangle + |1110\rangle)] \quad (44)$$
$$|\varphi_2^{\pm}\rangle = \frac{1}{2}[(|0101\rangle + |0110\rangle) \pm (|1000\rangle + |1111\rangle)], \quad (45)$$

then one can rewrite the combined state (43) as,

$$\begin{aligned}|\psi\rangle_a|Q5\rangle_{1234} &= \frac{1}{2}[|\varphi_1^+\rangle_{a123}(\alpha|0\rangle_4 + \beta|1\rangle_4) + |\varphi_1^-\rangle_{a123}(\alpha|0\rangle_4 - \beta|1\rangle_4) + \\ &\quad |\varphi_2^+\rangle_{a123}(\beta|0\rangle_4 + \alpha|1\rangle_4) + |\varphi_2^-\rangle_{a123}(\alpha|0\rangle_4 - \beta|1\rangle_4], \quad (46)\end{aligned}$$

or, one can use the basis vectors,

$$|\xi_1^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle \pm |1101\rangle) \quad (47)$$
$$|\xi_2^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0111\rangle \pm |1110\rangle) \quad (48)$$
$$|\xi_3^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0101\rangle \pm |1000\rangle) \quad (49)$$
$$|\xi_4^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0110\rangle \pm |1111\rangle), \quad (50)$$

then the combined state would be,

$$\begin{aligned}|\psi\rangle_a|Q5\rangle_{1234} &= \frac{1}{2\sqrt{2}}[(|\xi_1^+\rangle_{a123} + |\xi_2^+\rangle_{a123})(\alpha|0\rangle_4 + \beta|1\rangle_4) + (|\xi_1^-\rangle_{a123} + |\xi_2^-\rangle_{a123})(\alpha|0\rangle_4 - \beta|1\rangle_4) + \\ &\quad (|\xi_3^+\rangle_{a123} + |\xi_4^+\rangle_{a123})(\alpha|1\rangle_4 + \beta|0\rangle_4) + (|\xi_3^-\rangle_{a123} + |\xi_4^-\rangle_{a123})(\alpha|1\rangle_4 - \beta|0\rangle_4)]. \quad (51)\end{aligned}$$

Whatever the basis set, Alice uses to make von Neumann measurements, she needs to send two classical bit of information to Bob. She could encode the operations that Bob should apply. On receiving the classical information, Bob can complete the protocol.

Let us now consider the scenario, when Alice makes a three-particle followed by a one-particle measurement. Unlike the GHZ-state, here the state changes with permutation of the particles. The states obtained on permutation would also belong to the same SLOCC class. However, for different states, one would need different distribution of particles for the measurement. In this specific case, Alice needs to make a measurement on the particles a, 2, and 3 followed by that on particle 1.

For the three-particle measurement, Alice can use the basis,

$$
\begin{aligned}
|\omega_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|000\rangle \pm |101\rangle) && (52)\\
|\omega_2^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|001\rangle \pm |100\rangle) && (53)\\
|\omega_3^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|010\rangle \pm |111\rangle) && (54)\\
|\omega_4^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|011\rangle \pm |110\rangle). && (55)
\end{aligned}
$$

Then the combined state (43) can be written as

$$
\begin{aligned}
|\psi\rangle_a|Q5\rangle_{1234} = \; & \frac{1}{4}[(|\omega_1^+\rangle_{a23}|+\rangle_1 + |\omega_1^-\rangle_{a23}|-\rangle_1 + |\omega_4^+\rangle_{a23}|+\rangle_1 - |\omega_4^+\rangle_{a23}|-\rangle_1)(\alpha|0\rangle_4 + \beta|1\rangle_4) + \\
& (|\omega_1^+\rangle_{a23}|-\rangle_1 + |\omega_1^-\rangle_{a23}|+\rangle_1 + |\omega_4^-\rangle_{a23}|+\rangle_1 - |\omega_4^-\rangle_{a23}|-\rangle_1)(\alpha|0\rangle_4 - \beta|1\rangle_4) + \\
& (|\omega_2^+\rangle_{a23}|+\rangle_1 - |\omega_2^-\rangle_{a23}|-\rangle_1 + |\omega_3^+\rangle_{a23}|+\rangle_1 - |\omega_3^+\rangle_{a23}|-\rangle_1)(\alpha|1\rangle_4 + \beta|0\rangle_4) + \\
& (|\omega_2^-\rangle_{a23}|+\rangle_1 - |\omega_2^+\rangle_{a23}|-\rangle_1 + |\omega_3^-\rangle_{a23}|+\rangle_1 - |\omega_3^-\rangle_{a23}|-\rangle_1)](\alpha|0\rangle_4 - \beta|1\rangle_4)] \quad (56)
\end{aligned}
$$

In two-party situation in this scenario, using the basis (52)-(55) for the measurement and using two classical bits of information the protocol can be carried out. For the three-party situation, the three cbits of classical communication would be required. For the other scenarios, which we considered for the $|GHZ\rangle$ and $|\Omega\rangle$ states, one needs to explore further to find the feasibility of the use of this category of states.

3.2 Teleportation of two-qubit state

In this section, we consider the possibility of teleporting an unknown arbitrary two-qubit state. As we shall see that it would be possible with only a few quadripartite entangled states. However, sometime one would be able to teleport subclasses of the general state.

Let us consider the scenario where Alice has an unknown two-qubit state of the particles a and b,

$$|\psi\rangle_{ab} = \alpha|00\rangle_{ab} + \beta|01\rangle_{ab} + \gamma|10\rangle_{ab} + \delta|11\rangle_{ab}. \quad (57)$$

Here α, β, γ and δ are complex numbers. She wishes to transmit this state to Bob using a quadripartite entangled state. So Bob will have two of the four qubits and Alice will have the other two. As in the teleportation protocol, Alice can make von Neumann

measurement on her qubits and communicate her results to Bob using a classical channel. Bob then attempts to create this state using unitary transformations on his two qubits. As before, Alice may have many bases to choose from. There can also exist many scenarios. In the first scenario, Alice makes a four-particle measurement and communicates classically with Bob. In the second scenario, Alice makes a three-particle measurements, followed by one particle measurement and classical communication. In this second scenario, one could have two-party and three-party situations. In a third scenario, Alice may choose to make two suitable Bell measurements, instead of a four-particle measurement.

If one considers a product state of two Bell-states as a quantum resource, or the state $|\chi\rangle$ of Ref [28], then one can teleport an arbitrary unknown two-qubit state. We shall now consider various quantum entangled resources as in the last section. It is shown that the cluster state $|\Omega\rangle$ may also be used for an arbitrary two-qubit state, however for other considered quantum resources, only a sub-class can be teleported.

3.2.1 Teleportation of two-qubit state using $|GHZ\rangle$ state

The GHZ state is not a suitable entangled quantum resource for the teleportation of an arbitrary unknown two-qubit state. However, an entangled two-qubit state of the the following form can be teleported with the GHZ state (1),

$$|\psi_1\rangle_{ab} = \sigma_i\sigma_j(\alpha|00\rangle_{ab} + \beta|11\rangle_{ab}. \tag{58}$$

Here σ_i and σ_j are Pauli matrix operators, while α and β are complex numbers. We can see this by rewriting the combined state of the six particles as:

$$\begin{aligned}
|\psi_1\rangle_{ab}|GHZ\rangle_{1234} &= \sigma_i\sigma_j(\alpha|0000\rangle_{ab12}|00\rangle_{34} + \alpha|0011\rangle_{ab12}|11\rangle_{34} + \\
&\quad \beta|1100\rangle_{ab12}|00\rangle_{34} + \beta|1111\rangle_{ab12}|11\rangle_{34} \\
&= |\pi_1^+\rangle_{ab12}(\alpha|00\rangle_{34} + \beta|11\rangle_{34}) + |\pi_1^-\rangle_{ab12}(\alpha|00\rangle_{34} - \beta|11\rangle_{34}) + \\
&\quad |\pi_2^+\rangle_{ab12}(\alpha|11\rangle_{34} + \beta|00\rangle_{34}) + |\pi_2^-\rangle_{ab12}(\alpha|11\rangle_{34} - \beta|00\rangle_{34}),
\end{aligned} \tag{59}$$

where $|\pi_{1,2}^\pm\rangle$ are the orthogonal states,

$$\begin{aligned}
|\pi_1^\pm\rangle &= \sigma_i\sigma_j|4GHZ_1^\pm\rangle \\
|\pi_2^\pm\rangle &= \sigma_i\sigma_j|4GHZ_3^\pm\rangle.
\end{aligned} \tag{60}$$

After Alice has made the measurement in this basis and conveyed the results to Bob using two classical bits, he can carry out unitary operations $(\sigma_i\sigma_0)(\sigma_j\sigma_0)$, $(\sigma_i\sigma_0)(\sigma_j\sigma_3)$, $(\sigma_i\sigma_1)(\sigma_j\sigma_1)$, or $(\sigma_i\sigma_1)(\sigma_j\sigma_2)$ for the corresponding results of $|\pi\rangle_1^+, |\pi\rangle_1^-, |\pi\rangle_2^+$ and $|\pi\rangle_2^-$. This way Bob can obtain the state (59).

Instead of making a four-particle measurement, Alice may choose to make a three-particle on the particles a, b, and 1, followed by a one-particle measurement on the particle 2. This will also result in the successful teleportation, except that now Alice will have to convey three classical bits of information to Bob. There is also a possibility of Alice making two Bell measurements on the particles a and 1, followed by on the particles b and 2. These situations need to be explored further.

3.2.2 Teleportation of two-qubit state using $|\Omega\rangle$ state

It turns out that the cluster state $|\Omega\rangle$ can be a quite important entangled quantum resource. An arbitrary two-qubit state can be teleported using this state. To see this we can rewrite the combined state of the systems given in (3) and (57) as

$$\begin{aligned}
|\psi\rangle_{ab}|\Omega\rangle_{1234} \quad = \quad & \frac{1}{2}[(\alpha|0000\rangle_{ab12}|01\rangle_{34} + \alpha|0001\rangle_{ab12}|11\rangle_{34} + \alpha|0010\rangle_{ab12}|00\rangle_{34} - \alpha|0011\rangle_{ab12}|10\rangle_{34}) + \\
& (\beta|0100\rangle_{ab12}|01\rangle_{34} + \beta|0101\rangle_{ab12}|11\rangle_{34} + \beta|0110\rangle_{ab12}|00\rangle_{34} - \beta|0111\rangle_{ab12}|10\rangle_{34}) + \\
& \gamma|1000\rangle_{ab12}|01\rangle_{34} + \gamma|1001\rangle_{ab12}|11\rangle_{34} + \gamma|1010\rangle_{ab12}|00\rangle_{34} - \gamma|1011\rangle_{ab12}|10\rangle_{34}) + \\
& \delta|1100\rangle_{ab12}|01\rangle_{34} + \delta|1101\rangle_{ab12}|11\rangle_{34} + \delta|1110\rangle_{ab12}|00\rangle_{34} - \delta|1111\rangle_{ab12}|10\rangle_{34})]. \quad (61)
\end{aligned}$$

Let us know consider the possibility of Alice making a four-particle measurement and then conveying her results to Bob. Bob then makes suitable unitary transformation on his two qubits. It turns out that following set of basis vectors can serve as a suitable measurement basis,

$$|\Omega_1\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle) \quad (62)$$

$$|\Omega_2\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle - |1001\rangle + |1111\rangle) \quad (63)$$

$$|\Omega_3\rangle = \frac{1}{2}(|0000\rangle - |0110\rangle + |1001\rangle + |1111\rangle) \quad (64)$$

$$|\Omega_4\rangle = \frac{1}{2}(|0000\rangle - |0110\rangle - |1001\rangle - |1111\rangle) \quad (65)$$

$$|\Omega_5\rangle = \frac{1}{2}(|0001\rangle + |0111\rangle + |1000\rangle - |1110\rangle) \quad (66)$$

$$|\Omega_6\rangle = \frac{1}{2}(|0001\rangle + |0111\rangle - |1000\rangle + |1110\rangle) \quad (67)$$

$$|\Omega_7\rangle = \frac{1}{2}(|0001\rangle - |0111\rangle + |1000\rangle + |1110\rangle) \quad (68)$$

$$|\Omega_8\rangle = \frac{1}{2}(|0001\rangle - |0111\rangle - |1000\rangle - |1110\rangle) \quad (69)$$

$$|\Omega_9\rangle = \frac{1}{2}(|0010\rangle + |0100\rangle + |1011\rangle - |1101\rangle) \quad (70)$$

$$|\Omega_{10}\rangle = \frac{1}{2}(|0010\rangle + |0100\rangle - |1011\rangle + |1101\rangle) \quad (71)$$

$$|\Omega_{11}\rangle = \frac{1}{2}(|0010\rangle - |0100\rangle + |1011\rangle + |1101\rangle) \quad (72)$$

$$|\Omega_{12}\rangle = \frac{1}{2}(|0010\rangle - |0100\rangle - |1011\rangle - |1101\rangle) \quad (73)$$

$$|\Omega_{13}\rangle = \frac{1}{2}(|0011\rangle + |0101\rangle - |1010\rangle + |1100\rangle) \quad (74)$$

$$|\Omega_{14}\rangle = \frac{1}{2}(|0011\rangle + |0101\rangle + |1010\rangle - |1100\rangle) \quad (75)$$

$$|\Omega_{15}\rangle = \frac{1}{2}(|0011\rangle + |0101\rangle - |1010\rangle + |1100\rangle) \quad (76)$$

$$|\Omega_{16}\rangle = \frac{1}{2}(|0011\rangle - |0101\rangle - |1010\rangle - |1100\rangle). \quad (77)$$

We can then rewrite (61) as,

$$\begin{aligned}
|\psi\rangle_{ab}|\Omega\rangle_{1234} = \; & \frac{1}{4}[|\Omega_1\rangle_{ab12}(\alpha|01\rangle_{34} + \beta|00\rangle_{34} + \gamma|11\rangle_{34} + \delta|10\rangle_{34}) \\
& +|\Omega_2\rangle_{ab12}(\alpha|01\rangle_{34} + \beta|00\rangle_{34} - \gamma|11\rangle_{34} - \delta|10\rangle_{34}) \\
& +|\Omega_3\rangle_{ab12}(\alpha|01\rangle_{34} - \beta|00\rangle_{34} + \gamma|11\rangle_{34} - \delta|10\rangle_{34}) \\
& +|\Omega_4\rangle_{ab12}(\alpha|01\rangle_{34} - \beta|00\rangle_{34} - \gamma|11\rangle_{34} + \delta|10\rangle_{34}) \\
& +|\Omega_5\rangle_{ab12}(\alpha|11\rangle_{34} - \beta|10\rangle_{34} + \gamma|01\rangle_{34} - \delta|10\rangle_{34}) \\
& +|\Omega_6\rangle_{ab12}(\alpha|11\rangle_{34} - \beta|10\rangle_{34} - \gamma|01\rangle_{34} + \delta|10\rangle_{34}) \\
& +|\Omega_7\rangle_{ab12}(\alpha|11\rangle_{34} + \beta|10\rangle_{34} + \gamma|01\rangle_{34} + \delta|10\rangle_{34}) \\
& +|\Omega_8\rangle_{ab12}(\alpha|11\rangle_{34} + \beta|10\rangle_{34} - \gamma|01\rangle_{34} - \delta|10\rangle_{34}) \\
& +|\Omega_9\rangle_{ab12}(\alpha|00\rangle_{34} + \beta|01\rangle_{34} - \gamma|10\rangle_{34} - \delta|11\rangle_{34}) \\
& +|\Omega_{10}\rangle_{ab12}(\alpha|00\rangle_{34} + \beta|01\rangle_{34} + \gamma|10\rangle_{34} + \delta|11\rangle_{34}) \\
& +|\Omega_{11}\rangle_{ab12}(\alpha|00\rangle_{34} - \beta|01\rangle_{34} - \gamma|10\rangle_{34} + \delta|11\rangle_{34}) \\
& +|\Omega_{12}\rangle_{ab12}(\alpha|00\rangle_{34} - \beta|01\rangle_{34} + \gamma|10\rangle_{34} - \delta|11\rangle_{34})
\end{aligned}$$

$$
\begin{aligned}
&-|\Omega_{13}\rangle_{ab12}(\alpha|10\rangle_{34} - \beta|11\rangle_{34} - \gamma|10\rangle_{34} + \delta|11\rangle_{34}) \\
&-|\Omega_{14}\rangle_{ab12}(\alpha|10\rangle_{34} - \beta|11\rangle_{34} + \gamma|10\rangle_{34} - \delta|11\rangle_{34}) \\
&-|\Omega_{15}\rangle_{ab12}(\alpha|10\rangle_{34} + \beta|11\rangle_{34} - \gamma|10\rangle_{34} - \delta|11\rangle_{34}) \\
&-|\Omega_{16}\rangle_{ab12}(\alpha|10\rangle_{34} + \beta|11\rangle_{34} + \gamma|10\rangle_{34} + \delta|11\rangle_{34})].
\end{aligned} \tag{78}
$$

Alice can now communicate four cbits of information to Bob. Bob can then recover the unknown state using appropriate unitary transformations. For example, if after the measurement the state of particles a,b,1, and 2 is $(\alpha|01\rangle + \beta|00\rangle + \gamma|11\rangle + \delta|10\rangle)$, then Bob performs $(\sigma_0 \otimes \sigma_1)$ operation on his particles to recover the general state (57).

There exist another scenario where Alice makes separate Bell measurement on (a,2) and (b,1) particles. To see what happens we can rewrite the state (61) as

$$
\begin{aligned}
|\psi\rangle_{ab}|\Omega\rangle_{1234} \;=\; & \frac{1}{4}[|\varphi^+\rangle_{a2}|\varphi^+\rangle_{b1}(\alpha|01\rangle_{34} + \beta|00\rangle + \gamma|11\rangle - \delta|10\rangle) + \\
& |\varphi^+\rangle_{a2}|\varphi^-\rangle_{b1}(\alpha|01\rangle_{34} - \beta|00\rangle + \gamma|11\rangle - \delta|10\rangle) + \\
& |\varphi^-\rangle_{a2}|\varphi^+\rangle_{b1}(\alpha|01\rangle_{34} + \beta|00\rangle - \gamma|11\rangle - \delta|10\rangle) + \\
& |\varphi^-\rangle_{a2}|\varphi^-\rangle_{b1}(\alpha|01\rangle_{34} - \beta|00\rangle_{34} - \gamma|11\rangle_{34} + \delta|10\rangle)_{34} + \\
& |\psi^+\rangle_{a2}|\varphi^+\rangle_{b1}(\alpha|11\rangle_{34} - \beta|10\rangle_{34} + \gamma|01\rangle_{34} + \delta|10\rangle_{34}) + \\
& |\psi^+\rangle_{a2}|\varphi^-\rangle_{b1}(\alpha|11\rangle_{34} + \beta|10\rangle_{34} + \gamma|01\rangle_{34} - \delta|10\rangle_{34}) + \\
& |\psi^-\rangle_{a2}|\varphi^+\rangle_{b1}(\alpha|11\rangle_{34} - \beta|10\rangle_{34} - \gamma|01\rangle_{34} - \delta|10\rangle_{34}) + \\
& |\psi^-\rangle_{a2}|\varphi^-\rangle_{b1}(\alpha|11\rangle_{34} + \beta|10\rangle_{34} - \gamma|01\rangle_{34} + \delta|10\rangle_{34}) + \\
& |\varphi^+\rangle_{a2}|\psi^+\rangle_{b1}(\alpha|00\rangle_{34} + \beta|01\rangle_{34} - \gamma|10\rangle_{34} + \delta|11\rangle_{34}) + \\
& |\varphi^+\rangle_{a2}|\psi^-\rangle_{b1}(\alpha|00\rangle_{34} - \beta|01\rangle_{34} - \gamma|10\rangle_{34} - \delta|11\rangle_{34}) + \\
& |\varphi^-\rangle_{a2}|\psi^+_{b1}\rangle(\alpha|00\rangle_{34} + \beta|01\rangle_{34} - \gamma|10\rangle_{34} + \delta|11\rangle_{34}) + \\
& |\varphi^-\rangle_{a2}|\psi^-\rangle_{b1}(\alpha|00\rangle_{34} - \beta|01\rangle_{34} + \gamma|10\rangle_{34} + \delta|11\rangle_{34}) + \\
& |\psi^+\rangle_{a2}|\psi^+\rangle_{b1}(-\alpha|10\rangle_{34} + \beta|11\rangle_{34} + \gamma|00\rangle_{34} + \delta|01\rangle_{34}) + \\
& |\psi^+\rangle_{a2}|\psi^-\rangle_{b1}(-\alpha|10\rangle_{34} - \beta|11\rangle_{34} + \gamma|00\rangle_{34} - \delta|01\rangle_{34}) + \\
& |\psi^-\rangle_{a2}|\psi^+\rangle_{b1}(-\alpha|10\rangle_{34} + \beta|11\rangle_{34} - \gamma|00\rangle 34 - \delta|01\rangle_{34}) + \\
& |\psi^-\rangle_{a2}|\psi^-\rangle_{b1}(-\alpha|10\rangle_{34} - \beta|11\rangle_{34} - \gamma|00\rangle_{34} + \delta|01\rangle_{34}).
\end{aligned} \tag{79}
$$

In this case, unitary operations by Bob on his individual qubits does not allow reconstruction of the original state. However, a controlled phase shift operation on particles 3

and 4, with the particle 3 as the control bit and 4 as the target bit, followed by a unitary operation yield the exact state. For example, if the measurement results of the particle (a,2) and (b,1) are $|\varphi^+\rangle$ and $|\varphi^+\rangle$, respectively the state of the particle collapse into the state $\alpha|01\rangle + \beta|00\rangle + \gamma|11\rangle - \delta|10\rangle$. This state can not be transformed into $|\psi\rangle_{ab}$ by individual unitary transformations only. A controlled phase operation transforms above state into $\alpha|01\rangle + \beta|00\rangle - \gamma|11\rangle - \delta|10\rangle$ which can be converted into $|\psi\rangle_{ab}$ by $(\sigma_3 \otimes \sigma_1)$. In the above protocol, use of the Bell measurement constraints perfect teleportation by nonlocal operation by Bob.

3.2.3 Teleportation of two qubit state using $|W\rangle$

A general unknown two-qubit state cannot be teleported using $|W\rangle$-state as a quantum resource. However, as in the case of the $|GHZ\rangle$-state as a quantum resource, an unknown state like $\alpha|00\rangle + \beta|11\rangle$ may be teleported if Bob could make entangled unitary transformations [30]. It may be possible to teleport a more general two-qubit state with a suitable state belonging to the W-state category under SLOCC. For example, probabilistic teleportation of the two-qubit state $\alpha|00\rangle + \beta|01\rangle + \beta|10\rangle$ is possible using the entangled resource as the prototype W-state.

3.2.4 Teleportation of two qubit state using $|Q4\rangle$

While arbitrary two-qubit state cannot be teleported with the $|Q4\rangle$ state, one may be able to teleport a subclass of states such as $\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle$ with suitable choice of basis and distribution of particles. Here Bob may need to use entangled unitary transformations.

3.2.5 Teleportation of two qubit state using $|Q5\rangle$

As before, the teleportation of an arbitrary unknown two-qubit state may not be possible. However, for a subclass of states, teleportation may be possible. For example, the state $\alpha|00\rangle + \beta|10\rangle + \gamma|11\rangle$, where the particles are distributed such that the particles 2 and 3 are with Alice and 1 and 4 are with Bob may be teleported if Bob could apply entangled unitary transformations.

3.3 Teleportation of three-qubit state

For the teleportation protocol, one last possibility that one may consider is an arbitrary and unknown three-qubit state $|\psi\rangle_{abc}$. In this scenario, Alice has access to one particle (1 or 2 or 3 or 4), while Bob has rest of the three particles. Alice performs a joint measurement on four particles with suitable basis and transmits her result to Bob who makes suitable unitary transformations on his three qubits.

It is clear that a quadripartite state would not be suitable for the teleportation of a general unknown three-qubit state. For the teleportation of such a state, one would need an entangled state of six qubits. For example, three Bell states together could be used for such a teleportation. Here Alice and Bob would share one qubit each of the three Bell pair. This quantum resource does not have genuine six-particle entanglement, but some other such states like cluster states may also work.

With quadripartite states, one would be able to teleport some subclasses of the the general three-qubit state. We illustrate this by giving a few examples of such subclasses.

3.3.1 Teleportation of three-qubit state using $|GHZ\rangle$

With this state as a quantum resource, for the teleportation we consider the following states:

$$|\psi_2\rangle_{abc} = \sigma_i\sigma_j\sigma_k(\alpha|000\rangle_{abc} + \beta|111\rangle_{abc}), \tag{80}$$

where $i, j, k = 0, 1, 2, 3$ and σ_i, σ_j and σ_k act on particle 1, 2 and 3 respectively.

The joint state of this three-qubit state and GHZ state can be written as,

$$\begin{aligned}
|\psi_2\rangle_{abc}|GHZ\rangle_{1234} &= \sigma_i\sigma_j\sigma_k(\alpha|0000\rangle_{abc1}|000\rangle_{234} + \alpha|0001\rangle_{abc1}|111\rangle_{234} + \\
&\quad \beta|1110\rangle_{abc1}|000\rangle_{234} + \beta|1111\rangle_{abc1}|111\rangle_{234}) \\
&= |\pi_3^+\rangle_{abc1}(\alpha|000\rangle_{234} + \beta|111\rangle_{234} + |\pi_3^-\rangle_{abc1}(\alpha|000\rangle_{234} - \beta|111\rangle)_{234} + \\
&\quad |\pi_4^+\rangle_{abc1}(\alpha|111\rangle_{234} + \beta|000\rangle_{234}) + |\pi_4^-\rangle_{abc1}(\alpha|111\rangle_{234} - \beta|000\rangle_{234})
\end{aligned} \tag{81}$$

Therefore if Alice performs a measurement in the basis,

$$|\pi_3^\pm\rangle = \frac{1}{\sqrt{2}}\sigma_i\sigma_j\sigma_k|0000\rangle \pm |1111\rangle) \tag{82}$$

$$|\pi_4^\pm\rangle = \frac{1}{\sqrt{2}}\sigma_i\sigma_j\sigma_k|0001\rangle \pm |1110\rangle) \tag{83}$$

and sends two classical bits of information to Bob, then Bob can apply unitary transformations $(\sigma_i\sigma_0)(\sigma_j\sigma_0)(\sigma_k\sigma_0), (\sigma_i\sigma_0)(\sigma_j\sigma_0)(\sigma_k\sigma_3), (\sigma_i\sigma_0)(\sigma_j\sigma_0)(\sigma_k\sigma_1), (\sigma_i\sigma_0)(\sigma_j\sigma_0)(\sigma_k\sigma_2)$ to retrieve the original three-qubit state.

3.3.2 Teleportation of three-qubit state using $|\Omega\rangle$

With this quantum resource, one teleport following subclass of two-qubit state:

$$|\psi_3\rangle_{abc} = \sigma_i\sigma_j(\alpha|\varphi^+\rangle_{ab}|1\rangle_c + \beta|\varphi^-\rangle_{ab}|0\rangle_c), \tag{84}$$

where $i, j = 0, 1, 2, 3$ and σ_i acts on the particles a or b, while σ_j act on particle c.

We can rewrite this combined state as:

$$\begin{aligned}
|\psi_3\rangle_{abc}(|0\rangle_1|\varphi^+\rangle_{23}|0\rangle_4 + |1\rangle_1|\varphi^-\rangle_{23}|1\rangle_4) &= \frac{1}{\sqrt{2}}\sigma_i\sigma_j[|\varphi^+\rangle_{ab}|10\rangle_{c1}\alpha|\varphi^+\rangle_{23}|0\rangle_4 + \\
&\quad |\varphi^+\rangle_{ab}|11\rangle_{c1}\alpha|\varphi^-\rangle_{23}|1\rangle_4 + |\varphi^-\rangle_{ab}|00\rangle_{c1}\beta|\varphi^+\rangle_{23}|0\rangle_4 \\
&\quad + |\varphi^-\rangle_{ab}|01\rangle_{c1}\beta|\varphi^-\rangle_{23}|1\rangle_4] \\
&= |\Omega_3^+\rangle_{abc1}(\alpha|\varphi^+\rangle_{23}|0\rangle_4 + \beta|\varphi^-\rangle_{23}|1\rangle_4) + \\
&\quad |\Omega_3^-\rangle_{abc1}(\alpha|\varphi^+\rangle_{23}|0\rangle_4 - \beta|\varphi^-\rangle_{23}|1\rangle_4) + \\
&\quad |\Omega_4^+\rangle_{abc1}(\alpha|\varphi^-\rangle_{23}|1\rangle_4 + \beta|\varphi^+\rangle_{23}|0\rangle_4) + \\
&\quad |\Omega_4^-\rangle_{abc1}(\alpha|\varphi^+\rangle_{23}|1\rangle_4 - \beta|\varphi^+\rangle_{23}|0\rangle_4),
\end{aligned} \tag{85}$$

If Alice makes the measurement in the basis

$$\begin{aligned}
|\Omega_3^\pm\rangle &= \frac{1}{\sqrt{2}}\sigma_i\sigma_j(|\varphi^+\rangle|00\rangle \pm |\varphi^-\rangle|11\rangle) \\
|\Omega_4^\pm\rangle &= \frac{1}{\sqrt{2}}\sigma_i\sigma_j(|\varphi^+\rangle|10\rangle \pm |\varphi^-\rangle|01\rangle),
\end{aligned} \tag{86}$$

then as we see, the state (84) can be teleported, once Alice sends two classical bits of information to Bob.

3.3.3 Teleportation of three-qubit state using $|W\rangle$

A three particle GHZ-state can be teleported with four particle GHZ state. Can the three-particle W-states be teleported with the four-particle W-state ? Let us consider general three particle W state

$$|\psi_4\rangle = \alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle + \delta|000\rangle \tag{87}$$

We can rewrite the combined state as,

$$\begin{aligned}
&|\psi_4\rangle_{abc}|W\rangle_{1234} \\
&= \alpha|0010\rangle_{abc1}|001\rangle_{234} + \alpha|0010\rangle_{abc1}|010\rangle_{234} + \alpha|0010\rangle_{abc1}|100\rangle_{234} + \alpha|0011\rangle_{abc1}|000\rangle_{234} + \\
&\quad \beta|0100\rangle_{abc1}|001\rangle_{234} + \beta|0100\rangle_{abc1}|010\rangle_{234} + \beta|0100\rangle_{abc1}|100\rangle_{234} + \beta|0101\rangle_{abc1}|000\rangle_{234} + \\
&\quad \gamma|1000\rangle_{abc1}|001\rangle_{234} + \gamma|1000\rangle_{abc1}|010\rangle_{234} + \gamma|1000\rangle_{abc1}|100\rangle_{234} + \gamma|1001\rangle_{abc1}|000\rangle_{234} + \\
&\quad \delta|0000\rangle_{abc1}|001\rangle_{234} + \delta|0000\rangle_{abc1}|010\rangle_{234} + \delta|0000\rangle_{abc1}|100\rangle_{234} + \delta|0001\rangle_{abc1}|000\rangle_{234}.
\end{aligned} \tag{88}$$

No orthogonal measurement would enable faithful teleportation of $|\psi_4\rangle$ state. However, if we consider the coefficients such as $\alpha = \beta = \gamma = \delta = \frac{1}{2}$ and Alice performs the measurement in the basis

$$\begin{aligned}
|\Sigma_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0010\rangle \pm |0011\rangle), \\
|\Sigma_2^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0100\rangle \pm |0101\rangle), \\
|\Sigma_3^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|1000\rangle \pm |1001\rangle), \\
|\Sigma_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|0000\rangle \pm |0001\rangle,
\end{aligned} \tag{89}$$

then Bob's three particle will in one of the states, $\frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |000\rangle)$ or $\frac{1}{2}(|001\rangle + |010\rangle + |100\rangle - |000\rangle)$. Here the second state can not be transformed into the original state by local unitary operation. However, a joint operation $|001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100| - |000\rangle\langle 000|$ could enable Bob to reproduce the original state. Here classical communication cost is one cbit.

4 Superdense coding

Superdense coding protocol has played an important role in the development of the field of Quantum Information. It was shown in [2] that using an entangled state as a resource, the classical capacity of a quantum channel can be enhanced. Suppose there are two parties -

Alice and Bob. If Alice sends a qubit to Bob, then Bob can extract only one classical bit of information. However, if they share a Bell state, then using the protocol, Alice can transmit two classical bits by sending one qubit. Below, we are studying this protocol in the context of quadripartite entangled states. We consider two scenarios. In the first scenario, single-receiver dense coding, there are only two parties - Alice and Bob. This is the conventional scenario. In the second scenario, multi-receiver dense coding, there are more than two parties. There is one sender and multiple receivers. If these receivers could make global operations, then this scenario would reduce to the single-receiver scenario. Therefore these receivers are restricted to make only local operations and communicate classically with one another.

4.1 Single-receiver superdense coding

In this section, we discuss the superdense coding capacity of various entangled states where a sender, Alice, sends either one, two, or three qubits to a receiver, Bob. We use the notations DC_1, DC_2, or DC_3 respectively for the scenarios when one, two, or three qubits are sent. In DC_1, two parties Alice and Bob share one of the above four-qubit entangled states such that Alice possesses one particle, say 1, and Bob possesses three particles 2, 3, and 4. Similarly, in DC_2, Alice possesses two particles 1 and 2, while Bob possesses 3 and 4. In DC_3, Alice has three particles 1, 2, and 3, whereas Bob has particle 4. For all such distributions, they follow the standard superdense coding protocol to transmit a classical message from Alice to Bob. In this protocol, Alice applies unitary operations $I, \sigma_1, i\sigma_2, \sigma_3$ with equal probabilities on her particles and sends them to Bob. Bob performs a joint measurement on all the four particles to retrieve the original message. It is to be noted that as some of the channels are asymmetric with respect to permutation of particles, the distribution may affect the superdense coding capacity of the states. But we will always consider the particle distribution for the superdense coding capacity to be maximal. Since orthogonal states can be perfectly distinguished with some suitable measurement basis, in principle the deterministic dense coding capacity mainly depends on how many orthogonal states are obtained by unitary encoding on Alice's side. Therefore, here we seek to find out the number, N, of orthogonal states obtained (amount of information is $log_2 N$) by unitary operations on the particles by the sender.

4.1.1 The $|GHZ\rangle$ state

Let us consider the case, when the shared resource state is the $|GHZ\rangle$ state. When Alice applies unitary operation on particle 1, it produces following states,

$$
\begin{aligned}
I \otimes I \otimes I \otimes I|GHZ\rangle &\to \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \\
\sigma_1 \otimes I \otimes I \otimes I|GHZ\rangle &\to \frac{1}{\sqrt{2}}(|1000\rangle + |0111\rangle) \\
i\sigma_2 \otimes I \otimes I \otimes I|GHZ\rangle &\to \frac{1}{\sqrt{2}}(|1000\rangle - |0111\rangle) \\
\sigma_3 \otimes I \otimes I \otimes I|GHZ\rangle &\to \frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle). \qquad (90)
\end{aligned}
$$

These states are mutually orthogonal and can be unambiguously distinguished. After receiving the qubit from Alice, Bob performs a joint four-particle von Neumann measurement in the four-particle basis $\{|4GHZ_1^{\pm}\rangle, |4GHZ_2^{\pm}\rangle\}$ to distinguish these states. In this way he acquires two bits of classical information by receiving only one qubit. Similarly in DC_2 scenario, Alice applies unitary operations on two particles. It gives rise to sixteen states out which eight are orthogonal. In this process Bob can access only three cbits - not four cbits - by receiving two qubit by measuring in a proper $\{|4GHZ_n^{\pm}\rangle\}$ basis. In DC_3 scenario, Alice applies unitary operations on her three qubits yielding sixteen orthogonal states. It leads to perfect transmission of four classical bits of information. In general using N particle GHZ state one may be able to send N bits of classical information by sending N-1 particles.

4.1.2 The $|W\rangle$ state

In the case of parties sharing a W-state, in the DC_1 scenario, unitary operations on qubit 1 give the following states,

$$
\begin{aligned}
I \otimes I \otimes I \otimes I|W\rangle &\to \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \\
\sigma_1 \otimes I \otimes I \otimes I|W\rangle &\to \frac{1}{2}(|1001\rangle + |1010\rangle + |1100\rangle + |0000\rangle) \\
i\sigma_2 \otimes I \otimes I \otimes I|W\rangle &\to \frac{1}{2}(-|1001\rangle - |1010\rangle - |1100\rangle + |1000\rangle) \\
\sigma_3 \otimes I \otimes I \otimes I|W\rangle &\to \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle - |1000\rangle). \qquad (91)
\end{aligned}
$$

These states are not orthogonal and unambiguously discrimination is not possible. Therefore, Bob's measurement would not allow him to perfectly distinguish these states and transmis-

sion of two cbits by sending one qubit can not be achieved with unit probability. However, four-qubit W-state $|W_{mn}\rangle$ can be used to send two cbits by transmitting one qubit.

In the DC_2 scenario, the $|W\rangle$ state can actually be useful. When Alice applies unitary transformation on two particles, it gives rise to following orthogonal states,

$$\begin{aligned}
I \otimes I \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle), \\
\sigma_3 \otimes \sigma_3 \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(|0001\rangle + |0010\rangle - |0100\rangle - |1000\rangle), \\
\sigma_1 \otimes I \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(|1001\rangle + |1010\rangle + |1100\rangle + |0000\rangle), \\
i\sigma_2 \otimes \sigma_3 \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(-|1001\rangle - |1010\rangle + |1100\rangle + |0000\rangle), \\
\sigma_3 \otimes \sigma_1 \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(|0101\rangle + |0110\rangle + |0000\rangle - |1100\rangle), \\
I \otimes i\sigma_2 \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(-|0101\rangle - |0110\rangle + |0000\rangle - |1100\rangle), \\
\sigma_1 \otimes i\sigma_2 \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(-|1101\rangle - |1110\rangle + |1000\rangle - |0100\rangle), \\
i\sigma_2 \otimes \sigma_1 \otimes I \otimes I|W\rangle &\rightarrow \frac{1}{2}(-|1101\rangle - |1110\rangle - |1000\rangle + |0100\rangle).
\end{aligned} \tag{92}$$

As these states are mutually orthogonal. Bob will be able to discriminate all these states with unit probability by von Neumann measurement and recover three cbits of information. The dense coding capacity for DC_3 scenario is limited to three cbits irrespective of conventional $|W\rangle$ state or $|W_{mn}\rangle$ as a quantum channel. It is interesting to note that a three particle W state $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ is unsuitable for teleportation of a qubit and also one can not send more than one cbit of information using the superdense coding protocol. But a four-particle W state of the above form though not suitable for a qubit teleportation, it can be used for superdense coding and one can transmit three cbits by sending two qubits. The peculiarity of this state is that more than one cbit or three cbits of information cannot be transmitted by sending one or three qubits respectively, but three cbits of information could be communicated with the transmission of two qubits.

4.1.3 The $|\Omega\rangle$ state

This state is the best quantum resource from the point of view of superdense coding. The classical information capacity in the DC_1 scenario is two cbits with the proper distribution of the qubits. However, in scenario DC_2, Alice can transmit four cbits by transmitting two

qubits. This makes the $|\Omega\rangle$ state as the best quantum resource of all the considered states. In this scenario, when Alice applies unitary transformations on particles 1 and 2, following sixteen orthogonal states are obtained:

$$
\begin{aligned}
I \otimes I \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle) \\
I \otimes \sigma_3 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|0000\rangle - |0110\rangle + |1001\rangle + |1111\rangle) \\
\sigma_3 \otimes I \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0110\rangle - |1001\rangle + |1111\rangle) \\
\sigma_3 \otimes \sigma_3 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|0000\rangle - |0110\rangle - |1001\rangle - |1111\rangle) \\
I \otimes \sigma_1 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|0100\rangle + |0010\rangle + |1101\rangle - |1011\rangle) \\
I \otimes \sigma_2 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(-|0100\rangle + |0010\rangle - |1101\rangle - |1011\rangle) \\
\sigma_3 \otimes \sigma_1 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|0100\rangle + |0010\rangle - |1101\rangle + |1011\rangle) \\
\sigma_3 \otimes \sigma_2 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(-|0100\rangle + |0010\rangle + |1101\rangle + |1011\rangle) \\
\sigma_1 \otimes I \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|1000\rangle + |1110\rangle + |0001\rangle - |0111\rangle) \\
\sigma_1 \otimes \sigma_3 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|1000\rangle - |1110\rangle + |0001\rangle + |0111\rangle) \\
\sigma_2 \otimes I \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(-|1000\rangle - |1110\rangle + |0001\rangle - |0111\rangle) \\
\sigma_2 \otimes \sigma_3 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(-|1000\rangle + |1110\rangle + |0001\rangle + |0111\rangle) \\
\sigma_1 \otimes \sigma_1 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|1100\rangle + |1010\rangle + |0101\rangle - |0011\rangle) \\
\sigma_1 \otimes \sigma_2 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(-|1100\rangle + |1010\rangle - |0101\rangle - |0011\rangle) \\
\sigma_2 \otimes \sigma_1 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(-|1100\rangle - |1010\rangle + |0101\rangle - |0011\rangle) \\
\sigma_2 \otimes \sigma_2 \otimes I \otimes I|\Omega\rangle &\rightarrow \frac{1}{2}(|1100\rangle - |1010\rangle - |0101\rangle - |0011\rangle)
\end{aligned}
\tag{93}
$$

Therefore, clearly the classical capacity in this scenario is four cbits which is the maximum possible. Thus, the $|\Omega\rangle$ state exhibits same information capacity as the tensor product of two Bell states. In DC_3 scenario, the classical information transmission is limited to four cbits only. With Quadripartite states, more than this capacity is not possible as the Hilbert space of four qubits is sixteen dimensional.

4.1.4 The $|Q4\rangle$ state

This state is not symmetric under the permutation of particles. Therefore, the success of the protocol depends on the distribution of the particles between the parties. For example, let Alice has the qubit 1, whereas Bob has the rest. Then on applying unitary operations on her qubit, the $|Q4\rangle$ can become,

$$\begin{aligned}
I \otimes I \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle) \\
\sigma_1 \otimes I \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|1000\rangle + |1101\rangle + |0000\rangle + |0110\rangle) \\
\sigma_2 \otimes I \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(-|1000\rangle - |1101\rangle + |0000\rangle + |0110\rangle) \\
\sigma_3 \otimes I \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0101\rangle - |1000\rangle - |1110\rangle).
\end{aligned} \tag{94}$$

These states are not orthogonal to each other, so the protocol does not succeed. However, if the particles are distributed such that Alice has the particle 2 and Bob has the particles 1, 3 and 4, then unitary operation on the particle 2 yield four orthogonal states,

$$\begin{aligned}
I \otimes I \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle) \\
I \otimes \sigma_1 \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0100\rangle + |0001\rangle + |1100\rangle + |1010\rangle) \\
I \otimes \sigma_2 \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(-|0100\rangle + |0001\rangle - |1100\rangle + |1010\rangle) \\
I \otimes \sigma_3 \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0101\rangle - |1000\rangle - |1110\rangle).
\end{aligned} \tag{95}$$

Therefore with this distribution of the particles, the protocol succeeds. In the scenario DC_2, Alice applies unitary operations on the qubits 1 and 2. It gives rise to eight orthogonal states,

$$\begin{aligned}
I \otimes I \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle) \\
I \otimes \sigma_3 \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle - |0101\rangle + |1000\rangle - |1110\rangle) \\
\sigma_3 \otimes I \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle + |0101\rangle - |1000\rangle - |1110\rangle) \qquad (96) \\
\sigma_3 \otimes \sigma_3 \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|0000\rangle - |0101\rangle - |1000\rangle + |1110\rangle) \\
\sigma_1 \otimes \sigma_1 \otimes I \otimes I|Q4\rangle &\rightarrow \frac{1}{2}(|1100\rangle + |1001\rangle + |0100\rangle + |0010\rangle)
\end{aligned}$$

$$\sigma_2 \otimes \sigma_1 \otimes I \otimes I|Q4\rangle \rightarrow \frac{1}{2}(-|1100\rangle - |1001\rangle + |0100\rangle + |0010\rangle)$$
$$\sigma_1 \otimes \sigma_2 \otimes I \otimes I|Q4\rangle \rightarrow \frac{1}{2}(-|1100\rangle + |1001\rangle - |0100\rangle + |0010\rangle)$$
$$\sigma_2 \otimes \sigma_2 \otimes I \otimes I|Q4\rangle \rightarrow \frac{1}{2}(|1100\rangle - |1001\rangle - |0100\rangle + |0010\rangle). \tag{97}$$

Therefore the dense coding capacity in DC_2 scenario is three cbits. The DC_3 scenario, the classical capacity of the $|Q4\rangle$ state is only three cbits, independent of particle distribution and choice of coefficients.

4.1.5 The $|Q5\rangle$ state

As in the case of the $|Q4\rangle$ state, the distribution of the particles is important. It turns out that the protocol can be implemented in the DC_1 scenario. Alice can transmit two cbits by sending one qubit when particles are distributed such that Alice has the particle 2 and Bob has the particles 1, 3 and 4. In this scenario, when Alice applies unitary operations following orthogonal states are formed:

$$I \otimes I \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle)$$
$$I \otimes \sigma_1 \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(|0100\rangle + |1111\rangle + |1001\rangle + |1010\rangle)$$
$$I \otimes i\sigma_2 \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(-|0100\rangle - |1111\rangle + |1001\rangle + |1010\rangle)$$
$$I \otimes \sigma_3 \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(|0000\rangle + |1011\rangle - |1101\rangle - |1110\rangle). \tag{98}$$

In the DC_2 scenario, when Alice applies unitary transformations on the qubits 1 and 2, eight orthogonal states are obtained:

$$I \otimes I \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle)$$
$$\sigma_1 \otimes I \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(|1000\rangle + |0011\rangle + |0101\rangle + |0110\rangle)$$
$$I \otimes \sigma_1 \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(|0100\rangle + |1111\rangle + |1001\rangle + |1010\rangle)$$
$$\sigma_1 \otimes \sigma_1 \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(|1100\rangle + |0111\rangle + |0001\rangle + |0010\rangle)$$
$$I \otimes \sigma_2 \otimes I \otimes I|Q5\rangle \rightarrow \frac{1}{2}(-|0100\rangle - |1111\rangle + |1001\rangle + |1010\rangle)$$

$$\begin{aligned}
\sigma_1 \otimes \sigma_2 \otimes I \otimes I|Q5\rangle &\to \frac{1}{2}(-|1100\rangle - |0111\rangle + |0001\rangle + |0010\rangle) \\
I \otimes \sigma_3 \otimes I \otimes I|Q5\rangle &\to \frac{1}{2}(|0000\rangle + |1011\rangle - |1101\rangle - |1110\rangle) \\
\sigma_1 \otimes \sigma_3 \otimes I \otimes I|Q5\rangle &\to \frac{1}{2}(|1000\rangle + |0011\rangle - |0101\rangle - |0110\rangle)
\end{aligned} \tag{99}$$

Therefore in this scenario, Alice can transmit three cbits by sending two qubits. One can also verify that in the DC_3 scenario, four cbits can be sent by Alice to Bob by the transmission of three qubits.

4.2 Multi-receiver superdense coding

As noted earlier, in this protocol, there is one sender, but more than one receiver. Alice wishes to transmit classical information to one of the receivers with the assistance of the other receiver. Therefore, this protocol could also be called assisted superdense coding. Consider the situation where Alice shares an entangled state with the receivers B_1 and B_2. If B_1 and B_2 combine together and make a global measurement then the situation is similar to that discussed above for a single-receiver. However in the multi-receiver scenario, B_1 and B_2 perform measurements locally instead of global measurements and are allowed to communicate through a classical channel. In this subsection, we examine the usefulness of the considered quadripartite states for this protocol.

In the superdense coding, Alice can convert the shared entangled state to a set of orthogonal states by applying suitable unitary operations on her particles. In this scenario, the task of B_1 and B_2 is to distinguish these orthogonal states by local operations and classical communications (LOCC). The subject of distinguishing the orthogonal states using LOCC has been discussed in the literature [33, 34, 35]. We would use these results.

Walgate et al [33] have shown that any *two* orthogonal multipartite states can be distinguished by LOCC. Walgate and Hardy [34] generalized this result to a system of a qubit and a qudit. They showed that if Alice has the qubit and she goes first, then a set of ℓ orthogonal states $|\psi^i\rangle$ is LOCC distinguishable iff there is a basis $\{|0\rangle_A, |1\rangle_A\}$ for Alice to make measurement such that in this basis

$$|\psi^i\rangle = |0\rangle_A|\eta_0^i\rangle_B + |1\rangle_A|\eta_1^i\rangle_B, \tag{100}$$

where $\langle\eta_0^i|\eta_0^j\rangle = \langle\eta_1^i|\eta_1^j\rangle = 0$ if $i \neq j$. These indices take the values from 1 to ℓ. Chen and Li [35] generalized this result to the case of more general systems and found a condition for

the LOCC distinguishability of a set of orthogonal states. In particular, they showed that ℓ orthogonal states $\{|\Psi_i\rangle\}$ is perfectly distinguishable by LOCC if there exists a set of product vectors such that each state $|\Psi_i\rangle$ is superposition of some of these product vectors as follows

$$|\Psi_i\rangle = |\Phi_i^1\rangle_A|\xi_i^1\rangle_B + ... + |\Phi_i^{m^i}\rangle_A|\xi_i^{m^i}\rangle_B, \tag{101}$$

and each product vector $|\Phi_i^{k^i}\rangle_A|\xi_i^{k^i}\rangle_B(1 \leq k^i \leq m^i)$ belongs to only a state $|\Psi_i\rangle$, i.e.,

$$\begin{aligned}\langle\Phi_i^{k^i}|\langle\xi_i^{k^i}|\Psi_j\rangle &= 0\,\forall\, i \neq j,\\ \langle\Phi_i^{k^i}|\langle\xi_i^{k^i}|\Psi_i\rangle &\neq 0.\end{aligned} \tag{102}$$

In other words if a set of multipartite possible states is LOCC distinguishable, each possible state can be written as linear combination of product vectors such that each product vector of a possible state is orthogonal to the other possible states. From above theorem one can infer that in $2 \otimes 2 \otimes 2 \otimes 2$ systems, for bipartite splitting the number of orthogonal states that can be perfectly distinguished by LOCC is bounded by the number of product states in the linear combination. For sixteen orthogonal states to be LOCC distinguishable these have to be all product states. Similarly at most four and eight orthogonal states can be distinguished perfectly by LOCC if the set of states are linear combination of four and two product states respectively.

4.2.1 The $|GHZ\rangle$ state

Now consider the case when the parties share a four particle GHZ-state. Here Alice wishes to convey classical information to B_2 (Bob-2) with the assistance of B_1 (Bob-1). There are a number of ways to distribute qubits among Alice, B_1 and B_2. Alice could have one or two qubits. If Alice has only one qubit, then she can convert the shared quadripartite state into a set of four orthogonal states. As is the DC$_1$ scenario, in that case Alice would be able to transmit only two cbits to B_2, with the assistance of B_1. Here we consider the case when Alice possess particles 1 and 2 and perform unitary operations on these with equal probabilities. She sends the particle 1 to B_1 and particle 2 to B_2. Then B_1 and B_2 share the eight orthogonal states,

$$|4GHZ_1^{\pm}\rangle_{1234} = \frac{1}{\sqrt{2}}(|00\rangle_{13}|00\rangle_{24} \pm |11\rangle_{13}|11\rangle_{24}),$$
$$|4GHZ_2^{\pm}\rangle_{1234} = \frac{1}{\sqrt{2}}(|01\rangle_{13}|11\rangle_{24} \pm |10\rangle_{13}|00\rangle_{24}),$$

$$|4GHZ_3^{\pm}\rangle_{1234} = \frac{1}{\sqrt{2}}(|01\rangle_{13}|01\rangle_{24} \pm |10\rangle_{13}|10\rangle_{24}),$$
$$|4GHZ_4^{\pm}\rangle_{1234} = \frac{1}{\sqrt{2}}(|11\rangle_{13}|01\rangle_{24} \pm |00\rangle_{13}|10\rangle_{24}). \tag{103}$$

These states can be written in product decomposition using Bell basis,

$$\begin{aligned}
|4GHZ_1^{+}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|\phi^{+}\rangle_{13}|\phi^{+}\rangle_{24} + |\phi^{-}\rangle_{13}|\phi^{-}\rangle_{24}),\\
|4GHZ_1^{-}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|\phi^{+}\rangle_{13}|\phi^{-}\rangle_{24} + |\phi^{-}\rangle_{13}|\phi^{+}\rangle_{24}),\\
|4GHZ_2^{+}\rangle_{1234} &= \frac{1}{\sqrt{2}}(-|\psi^{+}\rangle_{13}|\phi^{+}\rangle_{24} + |\psi^{-}\rangle_{13}|\phi^{-}\rangle_{24}),\\
|4GHZ_2^{-}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|\psi^{+}\rangle_{13}|\phi^{-}\rangle_{24} + |\psi^{-}\rangle_{13}|\phi^{+}\rangle_{24}),\\
|4GHZ_3^{+}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|\psi^{+}\rangle_{13}|\psi^{+}\rangle_{24} + |\psi^{-}\rangle_{13}|\psi^{-}\rangle_{24}),\\
|4GHZ_3^{-}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|\psi^{+}\rangle_{13}|\psi^{-}\rangle_{24} + |\psi^{-}\rangle_{13}|\psi^{+}\rangle_{24}),\\
|4GHZ_4^{+}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|\phi^{+}\rangle_{13}|\psi^{+}\rangle_{24} - |\phi^{-}\rangle_{13}|\psi^{-}\rangle_{24}),\\
|4GHZ_4^{-}\rangle_{1234} &= \frac{1}{\sqrt{2}}(-|\phi^{+}\rangle_{13}|\psi^{-}\rangle_{24} - |\phi^{-}\rangle_{13}|\psi^{+}\rangle_{24}).
\end{aligned} \tag{104}$$

Here after receiving the qubit from the Alice, B_1 makes a measurement in the Bell basis and conveys his results to B_2. B_2 then also makes a measurement on his two qubits in the Bell basis. Depending on his results, he can distinguish all the above eight orthogonal states and thus decipher the three cbits of the information.

Let us now consider the another situation, where Alice sends her two qubits to B_2 and none to B_1. After Alice applies unitary transformation, the eight orthogonal states would be as in (103). For the sake of convenience, let us assume that Alice has the particles 1 and 2; B_1 has particle 3 and B_2 has the particle 4. After Alice sends her two qubits to B_2, B_2 would have the particles 1, 2, and 3. The suitable decomposition of the states (103) can be written as

$$\begin{aligned}
|4GHZ_1^{+}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|3GHZ_1^{+}\rangle_{123}|+\rangle_4 + |3GHZ_1^{-}\rangle_{123}|-\rangle_4),\\
|4GHZ_1^{-}\rangle_{1234} &= \frac{1}{\sqrt{2}}(|3GHZ_1^{+}\rangle_{123}|-\rangle_4 + |3GHZ_1^{-}\rangle_{123}|+\rangle_4),
\end{aligned}$$

$$
\begin{aligned}
|4GHZ_2^+\rangle_{1234} &= \frac{1}{\sqrt{2}}(|3GHZ_2^+\rangle_{123}|+\rangle_4 - |3GHZ_2^-\rangle_{123}|-\rangle_4),\\
|4GHZ_2^-\rangle_{1234} &= \frac{1}{\sqrt{2}}(-|3GHZ_2^+\rangle_{123}|-\rangle_4 + |3GHZ_2^-\rangle_{123}|+\rangle_4),\\
|4GHZ_3^+\rangle_{1234} &= \frac{1}{\sqrt{2}}(|3GHZ_3^+\rangle_{123}|+\rangle_4 - |3GHZ_3^-\rangle_{123}|-\rangle_4),\\
|4GHZ_3^-\rangle_{1234} &= \frac{1}{\sqrt{2}}(-|3GHZ_3^+\rangle_{123}|-\rangle_4 + |3GHZ_3^-\rangle_{123}|+\rangle_4),\\
|4GHZ_4^+\rangle_{1234} &= \frac{1}{\sqrt{2}}(|3GHZ_4^+\rangle_{123}|+\rangle_4 + |3GHZ_4^-\rangle_{123}|-\rangle_4),\\
|4GHZ_4^-\rangle_{1234} &= \frac{1}{\sqrt{2}}(|3GHZ_4^+\rangle_{123}|-\rangle_4 + |3GHZ_4^-\rangle_{123}|+\rangle_4). \qquad (105)
\end{aligned}
$$

After making a one-particle von Neumann measurement in the $\{|\pm\rangle\}$ basis, B_1 communicates his results to B_2 using one cbit. B_2 can now make three-particle von Neumann measurement using the $\{|3GHZ_1^{\pm}\rangle, |3GHZ_2^{\pm}\rangle, |3GHZ_2^{\pm}\rangle, |3GHZ_2^{\pm}\rangle\}$ basis and decipher the state. In this way, Alice can communicate three cbits to B_2 with the assistance of B_1. We notice that in this second scenario, B_1 has to use fewer cbits to communicate with the B_2. This is the advantage of the second scenario over the first.

4.2.2 The $|\Omega\rangle$ state

Let us now consider another quantum resource, the $|\Omega\rangle$ state. As in the case of GHZ-state, if Alice has the particle 1 only, she can convert the entangled resource to at most four orthogonal states, and thus can send two cbits to B_2 with the assistance of B_1. Let us again consider the situation where Alice has two qubits, whereas B_1 and B_2 have one each. By applying unitary transformations on qubits 1 and 2, Alice can transform the state into sixteen orthogonal states. However, unlike in the GHZ -state, B_1 and B_2 can distinguish only four orthogonal state using LOCC. This is because, the $|\Omega\rangle$ state is a superposition of four terms and the Hilbert space of four qubits is sixteen dimensional. The states that can be distinguished are

$$
\begin{aligned}
|\Omega\rangle_1 &= \frac{1}{2}(|00\rangle_{13}|00\rangle_{24} + |01\rangle_{13}|10\rangle_{24} + |10\rangle_{13}|01\rangle_{24} - |11\rangle_{13}|11\rangle_{24}),\\
|\Omega\rangle_5 &= \frac{1}{2}(|00\rangle_{13}|10\rangle_{24} + |01\rangle_{13}|00\rangle_{24} + |10\rangle_{13}|11\rangle_{24} - |11\rangle_{13}|01\rangle_{24}),\\
|\Omega\rangle_9 &= \frac{1}{2}(|10\rangle_{13}|00\rangle_{24} + |11\rangle_{13}|10\rangle_{24} + |00\rangle_{13}|01\rangle_{24} - |01\rangle_{13}|11\rangle_{24}), nonumber \qquad (106)\\
|\Omega\rangle_{13} &= \frac{1}{2}(|10\rangle_{13}|10\rangle_{24} + |11\rangle_{13}|00\rangle_{24} + |00\rangle_{13}|11\rangle_{24} - |01\rangle_{13}|01\rangle_{24}). \qquad (107)
\end{aligned}
$$

One can see that the measurement in basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ by B_1 and B_2 would enable them to distinguish the four states. So using the $|\Omega\rangle$ state two cbits of information can be communicated in multi-receiver scenario. In the scenario where Alice sends her both qubits to B_2, the result would be the same, i.e., Alice would be able to communicate only two cbits to B_2.

4.2.3 The $|W\rangle$ state

This state is like the $|\Omega\rangle$ state in the sense that it is also a superposition of four terms. Therefore conclusions for various scenarios would be the same. In particular, we consider the case when Alice applies unitary transformation on the qubits 1 and 2 of the shared W-state which give rise to eight orthogonal states given in (92). Alice then sends one qubit each to B_1 and B_2. The four states which can be LOCC distinguishable are

$$\begin{aligned}
|W_1\rangle_{1234} &= \frac{1}{2}(|00\rangle_{13}(|01\rangle + |10\rangle)_{24} + (|01\rangle + |10\rangle)_{13}|00\rangle_{24},\\
|W_2\rangle_{1234} &= \frac{1}{2}(|00\rangle_{13}(|01\rangle - |10\rangle)_{24} + (|01\rangle - |10\rangle)_{13}|00\rangle_{24},\\
|W_3\rangle_{1234} &= \frac{1}{2}(|10\rangle_{13}(|01\rangle + |10\rangle)_{24} + (|11\rangle + |00\rangle)_{13}|00\rangle_{24},\\
|W_4\rangle_{1234} &= \frac{1}{2}(-|10\rangle_{13}(|01\rangle - |10\rangle)_{24} + (-|11\rangle + |00\rangle)_{13}|00\rangle_{24}.
\end{aligned} \tag{108}$$

Writing these states in the Bell basis, we obtain

$$\begin{aligned}
|W_1\rangle_{1234} &= (|\phi^+\rangle_{13} + |\phi^-\rangle_{13})|\psi^+\rangle_{24} + |\psi^+\rangle_{13}(|\phi^+\rangle_{24} + |\phi^-\rangle_{24}),\\
|W_2\rangle_{1234} &= (|\phi^+\rangle_{13} + |\phi^-\rangle_{13})|\psi^-\rangle_{24} + |\psi^-\rangle_{13}(|\phi^+\rangle_{24} + |\phi^-\rangle_{24}),\\
|W_3\rangle_{1234} &= (|\psi^+\rangle_{13} - |\psi^-\rangle_{13})|\psi^+\rangle_{24} + |\phi^+\rangle_{13}(|\phi^+\rangle_{24} + |\phi^-\rangle_{24}),\\
|W_4\rangle_{1234} &= (|\psi^-\rangle_{13} - |\psi^+\rangle_{13})|\psi^-\rangle_{24} + |\phi^-\rangle_{13}(|\phi^+\rangle_{24} + |\phi^-\rangle_{24}).
\end{aligned} \tag{109}$$

These states satisfy the criteria to be LOCC distinguishable. B_1 would make a measurement in the Bell basis on his qubits and communicates the result to B_2. B_2 makes his own Bell measurement and thus obtains two cbits of the information.

4.2.4 The $|Q4\rangle$ state

As in the above two cases, Alice can transmit at most two cbits to B2 using the multi-receiver protocol. When Alice has two qubits 1 and 2, then on applying the unitary transformations,

she can convert the $|Q4\rangle$ state into eight orthogonal states, given in (96). However, it appears that there does not exist an straightforward way to distinguish even four of these orthogonal states by LOCC, because one cannot easily put these states in the (101) form.

4.2.5 The $|Q5\rangle$ state

The situation about this state is like the $|W\rangle$ state. Alice can convert the shared $|Q5\rangle$ state into eight orthogonal states, given in (99). However, only four states are LOCC distinguishable by B_1 and B_2. One such set is,

$$\begin{aligned}
|Q5_1\rangle &= \frac{1}{2}(|00\rangle_{13}|00\rangle_{24} + |11\rangle_{13}|01\rangle_{24} + |10\rangle_{13}|11\rangle_{24} + |11\rangle_{13}|10\rangle_{24}),\\
|Q5_2\rangle &= \frac{1}{2}(|10\rangle_{13}|00\rangle_{24} + |01\rangle_{13}|01\rangle_{24} + |00\rangle_{13}|11\rangle_{24} + |01\rangle_{24}|10\rangle_{24}),\\
|Q5_3\rangle &= \frac{1}{2}(|00\rangle_{13}|10\rangle_{24} + |11\rangle_{13}|11\rangle_{24} + |10\rangle_{13}|01\rangle_{24} + |11\rangle_{13}|00\rangle_{24}),\\
|Q5_4\rangle &= \frac{1}{2}(|10\rangle_{13}|10\rangle_{24} + |01\rangle_{13}|11\rangle_{24} + |00\rangle_{13}|01\rangle_{24} + |01\rangle_{13}|00\rangle_{24}).
\end{aligned} \tag{110}$$

B_1 makes von Neumann measurement in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and communicates his results to B_2. B_2 also makes the von Neumann measurement in the same basis and can distinguish the four states, thus obtaining the two cbits from Alice.

5 Conclusion

In this paper, we have considered a number of different genuine quadripartite entangled states as quantum resources for the teleportation and the superdense coding. For the teleportation protocol, we have examined the possibility of transmitting one-qubit, two-qubit and three-qubit unknown quantum states. Apart from the conventional scenario, we have also considered the multi-party scenarios and alternately the situations where Alice chooses to make a series of von Neumann measurements instead of one von Neumann measurement. For the superdense coding, we have considered the conventional single-receiver scenario as well as multi-receiver scenarios.

We find that the cluster state $|\Omega\rangle$ can be a very useful quantum resource. It can be used to teleport an arbitrary two-qubit unknown state. Using this state one can also transmit four classical bits by sending two qubits. In most of the other scenarios, this state is at least as

good a resource as any other; often it is a better resource. Only in multi-receiver scenario this state is less successful than the $|GHZ\rangle$ state. This is because the LOCC distinguishability criteria requires that the resource should have minimal number of terms. This may even indicate that the $|\Omega\rangle$ state has "stronger" entanglement. The $|GHZ\rangle$ state is the next useful resource. One can use this state and $|\Omega\rangle$ to transmit one-qubit state in all possible ways. However the state $|GHZ\rangle$ is not as useful as the $|\Omega\rangle$ state in the superdense coding and transmitting multiple-qubit states. Other quantum states, $|W\rangle$, $|Q_4\rangle$, and $|Q_5\rangle$ can also be useful resources in a number of scenarios. Their full utility needs to be investigated further. An interesting thread to explore will the use of entangled unitary transformations in the implementation of various quantum communication protocols. Another interesting avenue would be the use of higher-dimensional entangled states.

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